Fractional Topological Insulators in Three Dimensions

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Topological insulators can be generally defined by a topological field theory with an axion angle θ of 0 or π. In this work, we introduce the concept of fractional topological insulator defined by a fractional axion angle and show that it can be consistent with time reversal T invariance if ground state degeneracies are present. The fractional axion angle can be measured experimentally by the quantized fractional bulk magnetoelectric polarization P_3, and a "halved" fractional quantum Hall effect on the surface with Hall conductance of the form $\sigma_H = \frac{p}{q} \frac{e^2}{2h}$ with p, q odd. In the simplest of these states the electron behaves as a bound state of three fractionally charged "quarks" coupled to a deconfined non-Abelian $SU(3)$ "color" gauge field, where the fractional charge of the quarks changes the quantization condition of P_3 and allows fractional values consistent with T invariance.

Most states of quantum matter are classified by the symmetries they break. However, topological states of quantum matter [1] evade traditional symmetry-breaking classification schemes, and are rather described by topological field theories (TFT) in the low-energy limit. For the quantum Hall effect, the TFT is the $2 + 1$ dimensional Chern-Simons (CS) theory [2] with coefficient given by the quantized Hall conductance. In the noninteracting limit, the integer quantized Hall (IQH) conductance in units of $\frac{e^2}{h}$ is given by the TKNN invariant [3] or first Chern number. In the presence of strong correlations, one can also observe the fractional quantum Hall effect (FQHE), where the Hall conductance is quantized in rational multiples of $\frac{e^2}{h}$. In both cases however, these topological states can exist only in a strong magnetic field which breaks time reversal (T) symmetry.

More recently, T-invariant topological insulators (TI) have been studied extensively [1,4,5]. The TI state was first predicted theoretically in HgTe quantum wells, and observed experimentally [6–9] soon after. The theory of TI has been developed along two independent routes. Topological band theory identified $\mathbb{Z}_2$ topological invariants for noninteracting band insulators [8,10,11]. The TFT of T-invariant insulators was first developed in $4 + 1$ dimensions, where the CS term is naturally T invariant [12,13]. Dimensional reduction then gives the TFT for TI in $3 + 1$ and $2 + 1$ dimensions [14]. The TFT is generally valid for interacting systems, and describes the experimentally measurable quantized magnetoelectric response. The coefficient of the topological term, the axion angle θ, is constrained to be either 0 or π by T invariance. The TFT has been further developed in Refs. [15,16]. More recently, it has been shown that it reduces to the topological band theory in the noninteracting limit [17].

By analogy with the relation between the IQHE and FQHE, one is naturally led to the question whether there can exist a "fractional TI." In $2 + 1$ dimensions, an explicit wavefunction for the fractional quantum spin Hall state was first proposed in Ref. [9], and the edge theory was investigated in Ref. [18]. The T-invariant fractional topological state has also been constructed explicitly in $4 + 1$ dimensions [12]. Since T-invariant TI form a dimensional ladder in 4, 3, and 2 dimensions [14,19,20], it is natural to investigate the T-invariant TI in $3 + 1$ dimensions. Fractional states generally arise from strong interactions. Since topological band theory cannot describe such interactions, we formulate the general theory in terms of the TFT. The TI is generally described by the effective action $S_\theta = \frac{\theta}{2\pi} \frac{e^2}{2\pi} \int d^3x dt \mathbf{E} \cdot \mathbf{B}$, where $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic fields [14]. Under periodic boundary conditions, the partition function and all physical quantities are invariant under shifts of θ by multiples of 2π. Since $\mathbf{E} \cdot \mathbf{B}$ is odd under T, it appears that the only values of θ allowed by T are 0 or π mod 2π.

In this Letter, we show that there exist T-invariant insulating states in $3 + 1$ dimensions with $P_3 \equiv \frac{\theta}{2\pi}$ quantized in noninteger, rational multiples of $\frac{1}{2}$ of the form $P_3 = \frac{1}{2} \frac{p}{q}$ with p, q odd integers. The magnetoelectric polarization P_3 is defined by the response equation $\mathbf{P} = -\frac{\mathbf{B}}{2\pi}(P_3 + \text{const})$, where $\mathbf{B}$ is an applied magnetic field and $\mathbf{P}$ is the induced electric polarization. Such a fractionalized bulk topological quantum number leads to a fractional quantum Hall conductance of $\frac{p}{q} \frac{e^2}{2h}$ on the surface of the fractional TI. In contrast to the usual QHE in $2 + 1$ dimensions, the surface QHE does not necessarily exhibit edge states and thus cannot be directly probed by transport measurements. Alternatively, it can only be experimentally observed through probes which couple to each surface separately,

such as magneto-optical Kerr and Faraday rotation [14,16]. Generically, a slab of fractional TI can have different fractional Hall conductance on the top and bottom surfaces, which can be determined separately by combined Kerr and Faraday measurements, independent of nonuniversal properties of the material [21]. Our approach is inspired by the composite particle, or projective construction of FQH states [2,22–25]. The idea is to decompose the electron with charge e into $\mathcal{N}$ fractionally charged, fermionic "partons", which have a dynamics of their own. One considers the case that the partons form a known topological state, say a topological band insulator. When the partons are recombined to form the physical electrons, a new topological state of electrons emerges. In the FQH case, for example, the $\nu = \frac{1}{3}$ Laughlin state can be obtained by splitting each electron into $\mathcal{N} = 3$ partons of charge $\frac{e}{3}$. Each parton fills the lowest Landau level and forms a noninteracting $\nu = 1$ IQH state. Ignoring the exponential factors, the parton wave function is the Slater determinant IQH wave function $\Psi(\{z_i\}) \propto \prod_{i<j}(z_i - z_j)$, and the electron wave function is obtained by gluing three partons together, which leads to the Laughlin wave function $\Psi_{1/3}(\{z_i\}) \propto \prod_{i<j}(z_i - z_j)^3$. Similarly, in $3 + 1$ dimensions one can construct an interacting many-body wave function by gluing partons which are in a $\mathbb{Z}_2$ topological band insulator state. The parton ground state wave function $\Psi_1(\{\mathbf{r}_n s_n\})$ is a Slater determinant describing the ground state of a noninteracting TI Hamiltonian such as the lattice Dirac model [14], with $\{\mathbf{r}_n s_n\}$, $n = 1, \ldots, N$ the position and spin coordinates of the partons. The electron wave function is obtained by requiring the coordinates of all $\mathcal{N}_c$ partons forming the same electron to be the same [22],

$$\Psi_{\mathcal{N}_c}(\{\mathbf{r}_n s_n\}) = [\Psi_1(\{\mathbf{r}_n s_n\})]^{\mathcal{N}_c}. \quad (1)$$

Equation (1) is the $(3 + 1)$-dimensional generalization of the Laughlin wave function, and serves as a trial wave function for the simplest fractional TI phases we propose.

More generally, we can consider $\mathcal{N}_f$ different "flavors" of partons, with $\mathcal{N}_c^{(f)}$ partons of each flavor $f = 1, \ldots, \mathcal{N}_f$. This decomposition has to satisfy two basic rules. First, to preserve the fermionic nature of the electron, the total number of partons per electron must be odd [Fig. 1(a)],

$$\mathcal{N}_c^{(1)} + \mathcal{N}_c^{(2)} + \cdots + \mathcal{N}_c^{(\mathcal{N}_f)} = \text{odd}. \quad (2)$$

Second, if $q_f < e$ is the (fractional) charge of partons of flavor f, the total charge of the partons must add up to the electron charge e,

$$\mathcal{N}_c^{(1)}q_1 + \mathcal{N}_c^{(2)}q_2 + \cdots + \mathcal{N}_c^{(\mathcal{N}_f)}q_{\mathcal{N}_f} = e. \quad (3)$$

For instance, the $\nu = \frac{1}{3}$ Laughlin state described above corresponds to $\mathcal{N}_f = 1$, $\mathcal{N}_c^{(1)} = 3$, and $q_1 = \frac{e}{3}$, which satisfies both conditions. Here we consider that partons of each flavor f condense in a (generally different)
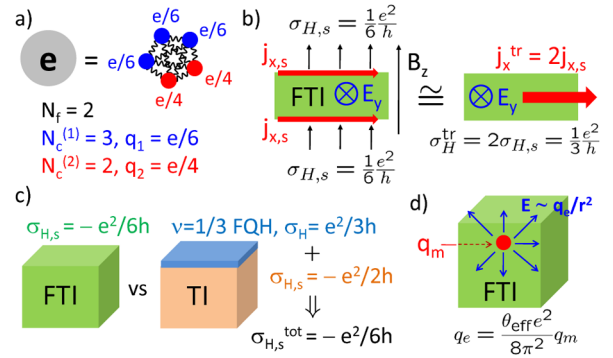

FIG. 1 (color online). (a) Quark picture of fractional TI with flavor and color degrees of freedom; (b) surface FQHE vs transport measurements [Eq. (7)]; (c) nontrivial vs trivial fractional TI; (d) Witten effect as a probe of bulk topology.

noninteracting T-invariant TI state with axion angle $\theta_f = \pi \mod 2\pi$. This is the analog of having partons condense in various IQH states in the FQH construction. Finally, the partons have to be bound together to yield physical electrons. As we will see, this can be done by coupling partons of flavor f to a $SU(\mathcal{N}_c^{(f)})$ gauge field, which can be interpreted as a "color field" where partons of flavor f come in $\mathcal{N}_c^{(f)}$ colors. Since the TI analog of the $\nu = \frac{1}{3}$ Laughlin state will involve three partons coupled to a $SU(3)$ gauge field in $3 + 1$ dimensions, we dub our partons "quarks" by analogy with quantum chromodynamics (QCD).

To obtain a more systematical understanding of the fractional TI, we now deduce its effective gauge theory by way of a gedanken experiment. We consider subjecting a noninteracting TI to strong electron-electron interactions, and start with the simplest case of $\mathcal{N}_f = 1$ with $\mathcal{N}_c$ odd. The electron being split into $\mathcal{N}_c$ quarks of charge $\frac{e}{\mathcal{N}_c}$, the electron operator will be written as a product of $\mathcal{N}_c$ quark operators $\psi_{i\alpha}$, $i = 1, \ldots, \mathcal{N}_c$. However, the quark operators act in a Hilbert space which is larger than the physical electron Hilbert space. We need to remove those states of the quark Hilbert space which are not invariant under unitary transformations which leave the electron operator unchanged, i.e., $SU(\mathcal{N}_c)$ transformations with quarks in the $\mathcal{N}_c$ representation. The projection onto the electron Hilbert space can therefore be implemented by coupling the quarks to a $SU(\mathcal{N}_c)$ gauge field a_μ with a coupling constant g. Outside the fractional TI, we expect the system to be in the confined phase, in analogy to quark confinement in QCD, which has only $SU(\mathcal{N}_c)$ singlet excitations in its low-energy spectrum, i.e., gauge-invariant "baryons". Quarks of a given flavor within the baryon are antisymmetric in their $\mathcal{N}_c^{(f)}$ color indices; Fermi statistics then implies that their spins are aligned. In a relativistic theory this would imply that in the $\mathcal{N}_f = 1$ theory the baryon has spin $\frac{\mathcal{N}_c}{2}$. In nonrelativistic lattice models this is not a concern, but even within the context of relativistic continuum field theories one can obtain composite spin-$\frac{1}{2}$ electrons for $\mathcal{N}_f > 1$.

Inside the fractional TI, electron-electron interactions translate into complicated interactions among quarks. We consider the case that these interactions lead the quarks to condense at low energies into a noninteracting T-invariant TI state with axion angle θ, and that the non-Abelian gauge field a_μ enters a deconfined phase [26]. We now show that such a phase is a fractional TI. A low-energy effective Lagrangian for $\mathcal{N}_f = 1$ can be conjectured in the form

$$\mathcal{L} = \psi^\dagger (iD_0 - H_\theta[-i\mathbf{D}])\psi + \mathcal{L}_{\text{int}}(\psi^\dagger, \psi), \quad (4)$$

where $D_\mu = (D_0, -\mathbf{D}) = \partial_\mu + i\frac{e}{\mathcal{N}_c}A_\mu + iga_\mu$ is the $U(1)_{\text{em}} \times SU(\mathcal{N}_c)$ gauge covariant derivative, and $H_\theta = H_\theta(\mathbf{p})$ is the single-particle Hamiltonian for a T-invariant TI with axion angle θ. $\mathcal{L}_{\text{int}}$ represents weak T-invariant residual interactions which do not destabilize the gapped TI phase, and can thus be safely ignored. The kinetic Yang-Mills Lagrangian for a_μ is generally present but not explicitly written.

Since the quarks are in a gapped TI phase, they can be integrated out to yield an effective Lagrangian for the gauge fields [14],

$$\mathcal{L}_{\text{eff}} = \frac{\theta}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}\text{Tr}\left(\frac{e}{\mathcal{N}_c}F_{\mu\nu} + gf_{\mu\nu}\right)\left(\frac{e}{\mathcal{N}_c}F_{\lambda\rho} + gf_{\lambda\rho}\right)$$
$$= \frac{\theta_{\text{eff}}e^2}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} + \frac{\theta g^2}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}\text{Tr}f_{\mu\nu}f_{\lambda\rho}, \quad (5)$$

where Tr is the trace in the $\mathcal{N}_c$ representation of $SU(\mathcal{N}_c)$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + ig[a_\mu, a_\nu]$ are the $U(1)_{\text{em}}$ and $SU(\mathcal{N}_c)$ field strengths, respectively, and the electromagnetic response is governed by an effective axion angle

$$\theta_{\text{eff}} = \frac{\theta}{\mathcal{N}_c} = 0, \pm\frac{\pi}{\mathcal{N}_c}, \pm\frac{3\pi}{\mathcal{N}_c}, \pm\frac{5\pi}{\mathcal{N}_c}, \ldots, \quad \mathcal{N}_c \text{ odd.} \quad (6)$$

Eq. (5) is obtained by replacing the $U(1)_{\text{em}}$ "electron" field strength $eF_{\mu\nu}$ in the $U(1)_{\text{em}}$ topological term $\frac{\theta}{2\pi}\frac{e^2}{2\pi}\mathbf{E}\cdot\mathbf{B} = \frac{\theta e^2}{32\pi^2}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$ for noninteracting TI [14] by the total $U(1)_{\text{em}} \times SU(\mathcal{N}_c)$ "quark" field strength $\frac{e}{\mathcal{N}_c}F_{\mu\nu} + gf_{\mu\nu}$. Note that the crossed terms of the form $\text{Tr}F_{\mu\nu}f_{\lambda\rho}$ vanish due to the tracelessness of the $SU(\mathcal{N}_c)$ gauge field. More generally, θ_{eff} can be obtained from the Adler-Bell-Jackiw anomaly, since θ corresponds to the phase of the quark mass [14]. In principle, the effective theory can also be obtained for quarks in a trivial insulator state with $\theta = 2n\pi$, $n \in \mathbb{Z}$. However, such a state is adiabatically connected to a trivial vacuum with $\theta = 0$, so that it is a trivial insulator in the bulk, although a fractional θ_{eff} can still be obtained due to pure surface effects. Since the focus of the present work is a fractional TI state with nontrivial bulk, we always consider quarks with $\theta = \pi$ mod 2π in the following.

We are now faced with our initial question of whether the effective theory (5) and (6) breaks T invariance. According to the first term in Eq. (5), T-invariance would

require θ_{eff} to be quantized in integer multiples of π if the minimal electric charge was e [27]. However, the minimal charge in our theory is $\frac{e}{\mathcal{N}_c}$, i.e., that of the quarks. Therefore, θ_{eff} has to be quantized in integer multiples of $\frac{\pi}{\mathcal{N}_c^2}$. On the other hand, the second term in Eq. (5) requires θ to be quantized in integer multiples of π [28], which means by Eq. (6) that θ_{eff} has to be quantized in units of $\frac{\pi}{\mathcal{N}_c}$. This latter constraint is consistent with, but stronger than, the former [29], and the values of θ_{eff} allowed by T invariance are thus correctly given by Eq. (6).

Eqs. (5) and (6) constitute a TFT which, precisely because it is topological, is insensitive to small T-invariant perturbations and defines a new stable phase of matter, the T-invariant fractional TI in $3 + 1$ dimensions. The effective theory can also be derived in the multiflavor case $\mathcal{N}_f \geq 1$, with $\mathcal{N}_c^{(f)}$ satisfying rules (2) and (3). Considering that quarks of flavor f form a noninteracting TI with axion angle $\theta_f = \pi$ mod 2π and integrating them out yields an effective Lagrangian in the form of (5), but with gauge group $U(1)_{\text{em}} \times \prod_{f=1}^{N_f} U(\mathcal{N}_c^{(f)})/U(1)_{\text{diag}}$. Here $U(1)_{\text{diag}}$ is the overall $U(1)$ gauge transformation of the electron operator. The electromagnetic axion angle θ_{eff} is given by $\theta_{\text{eff}} = (\sum_{f=1}^{\mathcal{N}_f}\frac{\mathcal{N}_c^{(f)}}{\theta_f})^{-1}$. When $\frac{\theta_f}{\pi}$ is odd for each flavor, one can show that $\theta_{\text{eff}} = \pi p/q$ with p, q odd integers.

Important physical properties of the fractional TI can be read off from Eq. (5). The surface of the fractional TI is an axion domain wall with the $U(1)_{\text{em}}$ axion angle jumping from θ_{eff} in the fractional TI to 0 in the vacuum. Such a domain wall has a surface QHE with surface Hall conductance $\sigma_{H,s} = \frac{\theta_{\text{eff}}}{2\pi}\frac{e^2}{h}$ [14]. Therefore, the surface Hall conductance of the fractional TI has the general form

$$\sigma_{H,s} = \frac{p}{q}\frac{e^2}{2h}, \qquad p, q \text{ odd.} \quad (7)$$

For example, in the simplest single-flavor case with $\theta = \pi$ in Eq. (5), we have $\sigma_{H,s} = \frac{1}{\mathcal{N}_c}\frac{e^2}{2h}$ with $\mathcal{N}_c$ an odd integer, corresponding to half of a $\nu = \frac{1}{\mathcal{N}_c}$ FQH Laughlin state. The more general result (7) corresponds to half of a generic Abelian FQH state [22,31].

The fractional axion angle and the associated surface Hall conductance (7) are properties of the bulk topology. It is important to distinguish them from a TI with $\theta_{\text{eff}} = \pm\pi$ and where the surface Dirac fermions form a FQH state [32]. In a noninteracting TI with $\theta_{\text{eff}} = -\pi$ for example, both the axion domain wall and the surface FQH state contribute to $\sigma_{H,s}$,

$$\sigma_{H,s} = \left(-\frac{1}{2} + \frac{n}{q}\right)\frac{e^2}{h} = \frac{2n-q}{q}\frac{e^2}{2h}, \quad (8)$$

with $\frac{n}{q}$ an allowed filling fraction for a FQH state in $2 + 1$ dimensions. For Abelian FQH states q is odd; hence the

surface Hall conductance $\sigma_{H,s} = \frac{2n-q}{q}\frac{e^2}{2h}$ has the same general form as for the fractional TI [Eq. (7)]. As the simplest example, the Laughlin state with $\frac{n}{q} = \frac{1}{3}$ leads to $\sigma_{H,s} = -\frac{1}{6}\frac{e^2}{h}$ [Fig. 1(c), right] which is the same as for a genuine fractional TI with bulk $P_3 = -\frac{1}{6}$ [Fig. 1(c), left]. However, the bulk topology is very different in both cases. Therefore, surface measurements are not sufficient to determine the bulk topology and bulk measurements of P_3 are needed. One such measurement would consist in embedding a monopole with magnetic charge q_m inside the fractional TI [Fig. 1(d)] and measuring its electric charge q_e induced by the Witten effect [33–35].

Another possible "experiment" is to measure the ground state degeneracy (GSD) on topologically nontrivial spatial 3-manifolds. Consider a fractional TI on a manifold $\Sigma_g \times I$ with Σ_g a Riemann surface of genus g and $I = [0, L]$ a bounded interval, where L is the sample "thickness" and the two copies of Σ_g (at each end of I) are the two bounding surfaces. We first discuss contributions to the GSD arising solely from the boundary, and comment on bulk contributions later on. A noninteracting TI with a $\nu = \frac{1}{3}$ Laughlin state deposited on both surfaces is described by two independent CS theories [2,36] and has a GSD of 3^g (m^g for $\nu = \frac{1}{m}$) on each surface for a total GSD of $(3^g)^2 = 3^{2g}$. The situation is different for a genuine fractional TI with $P_3 = \pm\frac{1}{6}$ [Fig. 1(c), left]. To study the GSD we set the external electromagnetic fields to zero in Eq. (5) and consider the internal $SU(3)$ θ term. Assuming that the system stays gapped as we take the limit of zero thickness $L \rightarrow 0$ where the gauge fields a_μ on both surfaces become identified, the system is described by a single $SU(3)_k$ CS theory on Σ_g where the level k is the sum of the contributions from both surfaces. If on both surfaces θ goes to the same value outside the TI, then $k = 0$ and there is no GSD. If $\theta = 0$ on one side and $\theta = 2\pi$ on the other, we have a $SU(3)_1$ CS theory with GSD $3^g \neq 3^{2g}$ [36]. In addition to the boundary contributions of the θ term to the GSD, the gauge theory in the bulk can have a nontrivial GSD even in the absence of boundaries. For instance, the deconfined phase of $SU(\mathcal{N}_c)$ gauge theory has a GSD of $\mathcal{N}_c^3$ on T^3 [37]. The total GSD has in general both bulk and boundary contributions, and depends on the details of the gauge group.

Note added.—After this Letter was originally posted, fractional TI were further investigated by B. Swingle *et al.* [38].

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