Dissipation-Induced *d*-Wave Pairing of Fermionic Atoms in an Optical Lattice

S. Diehl,[1,2] W. Yi,[1,3,*] A. J. Daley,[1,2] and P. Zoller[1,2]

[1]*Institute for Quantum Optics and Quantum Information of the Austrian Academy of Sciences, A-6020 Innsbruck, Austria*
[2]*Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria*
[3]*Key Laboratory of Quantum Information, University of Science and Technology of China,
CAS, Hefei, Anhui,230026, People's Republic of China*

We show how dissipative dynamics can give rise to pairing for two-component fermions on a lattice. In particular, we construct a parent Liouvillian operator so that a BCS-type state of a given symmetry, e.g., a *d*-wave state, is reached for arbitrary initial states in the absence of conservative forces. The system-bath couplings describe single-particle, number-conserving and quasilocal processes. The pairing mechanism crucially relies on Fermi statistics. We show how such Liouvillians can be realized via reservoir engineering with cold atoms representing a driven dissipative dynamics.

Pairing in condensed matter physics in general, and in atomic quantum gases in particular, is associated with conservative forces between particles, e.g., in Cooper pairs or molecular BEC pairs [1]. Lattice dynamics gives rise to exotic forms of pairing, such as the expected formation of *d*-wave Cooper pairs of fermions for a 2D Hubbard model for repulsive interactions, as discussed in the context of high-T_c superconductivity [2], but also condensates of η pairs [3], and the formation of repulsively bound atom pairs [4]. Here we show that purely dissipative dynamics, induced by coupling the system to a bath, can give rise to pairing, even in the complete absence of conservative forces. This "dissipative pairing" crucially relies on Fermi statistics and is in contrast to pairing arising from bath-mediated interactions (e.g., phonon-mediated Cooper pairing). We will discuss how reservoir engineering provides opportunities for experimental realization of this dissipative pairing mechanism with cold atomic fermions in optical lattices [5].

Below we treat the example of a *d*-wave-paired BCS state of two-component fermions in two dimensions (2D), showing how the pairing can be generated via purely *dissipative* processes. A BCS-type state is the conceptually simplest many-body wave function describing a condensate of N paired spin-1/2 fermionic particles, $|\text{BCS}_N\rangle \sim (d^\dagger)^{N/2}|\text{vac}\rangle$. On a square lattice, and assuming singlet pairs with zero center-of-mass momentum, we have $d^\dagger = \sum_{\mathbf{q}} \varphi_{\mathbf{q}} c^\dagger_{\mathbf{q},\uparrow} c^\dagger_{-\mathbf{q},\downarrow}$ or $d^\dagger = \sum_{i,j} \varphi_{ij} c^\dagger_{i,\uparrow} c^\dagger_{j,\downarrow}$, where $c^\dagger_{\mathbf{q},\sigma}$ ($c^\dagger_{i,\sigma}$) denotes the creation operator for fermions with quasimomentum $\mathbf{q}$ (on lattice site i) and spin $\sigma = \uparrow, \downarrow$, and $\varphi_{\mathbf{q}}$ (φ_{ij}) the momentum (position) wave function of the pairs. For *d*-wave pairing, the pair wave function obeys $\varphi_{q_x,q_y} = -\varphi_{-q_y,q_x} = \varphi_{-q_x,-q_y}$, and below we choose $\varphi_{\mathbf{q}} = \cos q_x - \cos q_y$ or $\varphi_{ij} = \frac{1}{2}\sum_{\lambda=x,y} \rho_\lambda(\delta_{i,j+\mathbf{e}_\lambda} + \delta_{i,j-\mathbf{e}_\lambda})$ with $\rho_x = -\rho_y = 1$ corresponding to the limit of well localized pairs [see Fig. 1(a)], and $\mathbf{e}_\lambda$ the unit lattice vector in $\lambda = x$, y direction. For reference below we remark that in BCS

theory, with pairing induced by *coherent* interactions, the corresponding energy gap function would be $\Delta_{\mathbf{q}} = \Delta(\cos q_x - \cos q_y)$ in the molecular limit. The dissipative pairing mechanism is readily generalized to other pairing symmetries, such as, e.g., $p_x + ip_y$ [6], as long as the pairing is not on site.

While in the standard scenario BCS-type states are typically used as variational mean-field wave functions to describe pairing due to interactions, here the system is dissipatively driven towards the (pure) many-body BCS state, $\rho(t) = e^{\mathcal{L}t}\rho(0) \overset{t\to\infty}{\longrightarrow} |\text{BCS}_N\rangle\langle\text{BCS}_N|$, beginning from an arbitrary initial mixed state $\rho(0)$. The dynamics of the density matrix for the N-particle system $\rho(t)$ is generated by a Liouville operator with the structure $\mathcal{L}\rho = -iH_{\text{eff}}\rho + i\rho H^\dagger_{\text{eff}} + \kappa\sum_\ell j_\ell \rho j^\dagger_\ell$ with non-Hermitian effective Hamiltonian $H_{\text{eff}} = H - \frac{i}{2}\kappa\sum_\ell j^\dagger_\ell j_\ell$. Here, $\{j_\ell\}$ are non-Hermitian Lindblad operators reflecting the system-bath coupling with strength characterized by the rate κ.
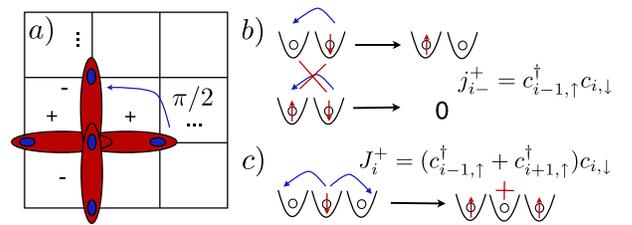
FIG. 1 (color online). (a) Symmetry in the *d*-wave state, represented by a single off site fermion pair exhibiting the characteristic sign change under spatial rotations. In a *d*-wave BCS state, this pair is delocalized over the whole lattice. (b), (c) The dissipative pairing mechanism builds on (b) Pauli blocking and (c) delocalization via phase locking. (b) Illustration of the action of Lindblad operators using Pauli blocking for a Néel state (see text). (c) The *d*-wave state may be seen as a delocalization of these pairs away from half-filling (shown is a cut along one lattice axis).

The Hamiltonian H generates unitary evolution, and will be set to zero for most of the discussion. The pure paired BCS state being the unique steady state of the dissipative dynamics results from the possibility to identify a set of operators with $j_\ell|\text{BCS}_N\rangle = 0 \; \forall \ell$ [7,8].

Below, we will identify these operators j_ℓ for the d-wave paired BCS states, and in addition study the dynamics close to the final steady state, i.e., near $|\text{BCS}_N\rangle$. We can then investigate the complex excitation spectrum of $\mathcal{L}$, where, remarkably, we find a dissipative "BCS gap" that implies exponential approach to the steady state.

We can readily check that the Lindblad operators j_ℓ generating the d-wave BCS state are given by

$$J_i^\alpha = \sum_{\lambda=x,y} \rho_\lambda (c_{i+\mathbf{e}_\lambda}^\dagger + c_{i-\mathbf{e}_\lambda}^\dagger) \sigma^\alpha c_i, \qquad (1)$$

with 2-spinor $c_i = (c_{i,\uparrow}, c_{i,\downarrow})^T$ and σ^α Pauli matrices with $\alpha = \pm, z$ or $\alpha = x, y, z$. An explicit construction is given below. Remarkably, these Lindblad operators, which generate pairing dissipatively, are bilinear and number conserving, thus acting on a *single particle* only. They are also quasilocal operators, involving only a plaquette of nearest neighbor sites [see Fig. 1(a)].

Before entering the more technical discussion of obtaining these J_i^α, we discuss the dynamics for states close to the final state $|\text{BCS}_N\rangle$, where the physics is particularly transparent and analogies to the usual case of interaction-induced pairing in BCS theory can be made. For states close to $|\text{BCS}_N\rangle$ we can linearize the master equation dynamics using a Bogoliubov-type approach. Here we take advantage of the fact that we know the steady state for our problem exactly; this knowledge can be used to construct a quadratic theory for the fluctuations on top of it. For this purpose it is technically convenient to give up exact particle number conservation, and to work with fixed phase coherent states $|\text{BCS}_\theta\rangle = \mathcal{N}^{-1/2}\exp(e^{i\theta}d^\dagger)|\text{vac}\rangle$ instead of the number states $|\text{BCS}_N\rangle$ [1], where $\mathcal{N} = \prod_{\mathbf{q}}(1 + \varphi_{\mathbf{q}}^2)$ ensures the normalization. The density matrix for these states, describing the dark steady state, factorizes in momentum space since $\exp(e^{i\theta}d^\dagger)|\text{vac}\rangle = \prod_{\mathbf{q}}(1 + e^{i\theta}\varphi_{\mathbf{q}}c_{\mathbf{q},\uparrow}^\dagger c_{-\mathbf{q},\downarrow}^\dagger)|\text{vac}\rangle$. At late times, we can therefore expand the state around $|\text{BCS}_\theta\rangle$ by making the factorized ansatz $\rho = \prod_{\mathbf{q}}\rho_{\mathbf{q}}$, where $\rho_{\mathbf{q}}$ contains the modes $\pm(\mathbf{q}, \sigma)$ necessary to describe pairing. We can then utilize the projection prescription $\rho_{\mathbf{q}} = \text{tr}_{\neq\mathbf{q}}\rho$ to find the equations of motion for the single pair density matrices $\rho_{\mathbf{q}}$ in the presence of nonzero mean fields. These result from the coupling to other momentum modes, and their values are dictated by the final state properties. The resulting effective Hamiltonian is quadratic:

$$H_{\text{eff}} = -\frac{i\kappa}{2}\sum_{\mathbf{q},\sigma}\{\tilde{n}(c_{\mathbf{q},\sigma}^\dagger c_{\mathbf{q},\sigma} + |\varphi_{\mathbf{q}}|^2 c_{\mathbf{q},\sigma} c_{\mathbf{q},\sigma}^\dagger)$$
$$+ \tilde{\Delta}_{\mathbf{q}} s_\sigma c_{-(\mathbf{q},\sigma)} c_{\mathbf{q},\sigma} + \text{H.c.}\}$$
$$= -\frac{i}{2}\sum_{\mathbf{q},\sigma}\kappa_{\mathbf{q}}\gamma_{\mathbf{q},\sigma}^\dagger \gamma_{\mathbf{q},\sigma}, \qquad (2)$$

with $s_\uparrow = -1$, $s_\downarrow = 1$ and dimensionless "gap function" $\tilde{\Delta}_{\mathbf{q}} = \tilde{\Delta}\varphi_{\mathbf{q}}$, and where the diagonal and off diagonal mean fields evaluate to $\tilde{n} = |\tilde{\Delta}| = 2\int\frac{d\mathbf{q}}{(2\pi)^2}\frac{|\varphi_{\mathbf{q}}|^2}{1+|\varphi_{\mathbf{q}}|^2} \approx 0.72$ on the d-wave state, where the integration is over the Brillouin zone. We diagonalize H_{eff} in the second line, introducing quasiparticle Lindblad operators

$$\gamma_{\mathbf{q},\sigma} = (1 + \varphi_{\mathbf{q}}^2)^{-1/2}(c_{-\mathbf{q},\sigma} + s_\sigma \varphi_{\mathbf{q}} c_{\mathbf{q},-\sigma}^\dagger).$$

In this basis, the resulting master equation reads $\partial_t\rho = -iH_{\text{eff}}\rho + i\rho H_{\text{eff}}^\dagger + \sum_{\mathbf{q},\sigma}\kappa_{\mathbf{q}}\gamma_{\mathbf{q},\sigma}\rho\gamma_{\mathbf{q},\sigma}^\dagger$. The linearized Lindblad operators have analogous properties to quasiparticle operators familiar from interaction pairing problems: (i) They annihilate the (unique) steady state $\gamma_{\mathbf{q},\sigma}|\text{BCS}_\theta\rangle = 0$; (ii) they obey the Dirac algebra $\{\gamma_{\mathbf{q},\sigma}, \gamma_{\mathbf{q}',\sigma'}^\dagger\} = \delta_{\mathbf{q},\mathbf{q}'}\delta_{\sigma,\sigma'}$ and zero otherwise [9]; and (iii) therefore are related to original fermions via a canonical transformation. The imaginary spectrum of the effective Hamiltonian features a "dissipative pairing gap"

$$\kappa_{\mathbf{q}} = \kappa\tilde{n}(1 + \varphi_{\mathbf{q}}^2) \geq \kappa\tilde{n}.$$

The dissipative gap implies an exponential approach to the steady d-wave BCS state for long times. This can be most easily seen in a quantum trajectory representation of the master equation, where the system's time evolution is described by a stochastic wave function $|\psi(t)\rangle$ evolving under a non-Hermitian Hamiltonian $|\psi(t)\rangle = e^{-iH_{\text{eff}}t}|\psi(0)\rangle/\|\cdots\|$ interrupted with rate $\kappa\||j_\ell|\psi(t)\rangle\|^2$ by quantum jumps $|\psi(t)\rangle \rightarrow j_\ell|\psi(t)\rangle/\|\cdots\|$ so that $\rho(t) = \langle|\psi(t)\rangle \times \langle\psi(t)|\rangle_{\text{stoch}}$ (see, e.g., Ref. [10]). We thus see that (i) the BCS state is a "dark state" of the dissipative dynamics in the sense that $j_\ell|\text{BCS}_N\rangle = 0$ implies that there will never be a quantum jump, i.e., the state remains in $|\text{BCS}_N\rangle$, and (ii) states near $|\text{BCS}_N\rangle$ show an exponential decay according to the dissipative gap. This is in marked contrast to dissipative preparation of a noninteracting BEC in bosonic systems, where the approach is polynomial in time [7].

This convergence to a unique pure state is illustrated in Fig. 2 using numerical simulations for small systems. In Fig. 2(a) we show the entropy of the full density matrix for a small 1D system as a function of time, and in Fig. 2(b) the fidelity of the BCS state for a small 2D grid, computed via the quantum trajectories method.

Lindblad operators for d-wave states.—We now turn to the construction of the Lindblad operators for the d-wave BCS state as given in Eq. (1). We will perform this construction first for an antiferromagnetic Néel state at half-filling, and then generalize to the BCS state. For a given many-body state $|d\rangle$, we require a set of (non-Hermitian) Lindblad operators j_ℓ so that it becomes the unique dark state, $j_\ell|d\rangle = 0 \; \forall l$. Both the Néel and the BCS state have product form, $|d\rangle = \prod_m d_m^\dagger|\text{vac}\rangle$. Thus, we note as a sufficient dark state condition $[j_\ell, d_m^\dagger] = 0$.

There are two antiferromagnetic Néel states at half-filling $|N+\rangle = \prod_{i\in A}c_{i+\mathbf{e}_x,\uparrow}^\dagger c_{i,\downarrow}^\dagger|\text{vac}\rangle$, $|N-\rangle = \prod_{i\in A}c_{i+\mathbf{e}_x,\downarrow}^\dagger c_{i,\uparrow}^\dagger|\text{vac}\rangle$ with A a sublattice in a two-dimensional bipartite (square)
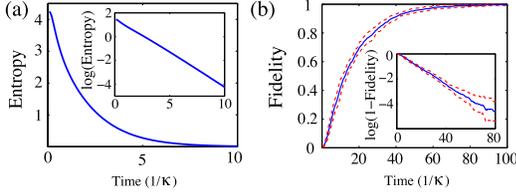
FIG. 2 (color online). Numerical illustration of the uniqueness of the steady state, showing evolution under the master equation with Lindblad operators from Eq. (1). (a) Entropy computed exactly for four atoms on a 4×1 lattice, showing exponential convergence from a completely mixed state to a pure state. (b) Fidelity to the d-wave BCS state, $\langle BCS_N | \rho | BCS_N \rangle$ with 4 atoms on a 4×4 grid, computed via a quantum trajectories method (see text). Dashed lines show sampling error.

lattice, which differ by an overall spin flip. Introducing "Néel unit cell operators" $\hat{S}_{i,\nu}^a = c_{i+\mathbf{e}_\nu}^\dagger \sigma^a c_i^\dagger$, $a = \pm$, $\mathbf{e}_\nu = \{\pm\mathbf{e}_x, \pm\mathbf{e}_y\}$, whose usefulness will become apparent soon, the state can be written in eight different forms, $|N\pm\rangle = \prod_{i\in A} \hat{S}_{i,\nu}^\pm |\mathrm{vac}\rangle = (-1)^{M/2} \prod_{i\in B} \hat{S}_{i,-\nu}^\mp |\mathrm{vac}\rangle$, with M the lattice size. We then see that the Lindblad operators must obey $[j_{i,\nu}^a, \hat{S}_{j,\mu}^b] = 0$ for all i, j located on the same sublattice A or sublattice B, which is fulfilled for the set

$$j_{i,\nu}^a = c_{i+\mathbf{e}_\nu}^\dagger \sigma^a c_i, \qquad i \in A \quad \text{or} \quad B. \qquad (3)$$

Note that these operators can be obtained from $\hat{S}_{i,\nu}^a$ by a particle-hole transformation $c_{i,\sigma}^\dagger \rightarrow c_{i,\sigma}$ on the central site i. For the action of the operators $j_{i,\nu}^a$ the assumption of fermionic statistics is essential, as illustrated in Fig. 1(b): they generate spin flipping transport according to, e.g., $j_{i,\nu}^+ = c_{i+\mathbf{e}_\nu,\uparrow}^\dagger c_{i,\downarrow}$, which is not possible when the antiferromagnetic order is already present. The proof of uniqueness of the Néel steady state up to double degeneracy is then trivial: The steady state must fulfill the quasilocal condition that for any site occupied by a certain spin, its neighboring sites be filled by opposite spins. For half-filling, the only states with this property are $|N\pm\rangle$. The residual degeneracy can be lifted by adding a single operator $j_i = c_{i+\mathbf{e}_\nu}^\dagger (1 + \sigma^z) c_i$ at arbitrary i.

To find the Lindblad operators for the d-wave BCS state, we apply a similar strategy. We first rewrite the d-wave generator using the operators $\hat{S}_i^a$,

$$d^\dagger = \frac{i}{2} \sum_i (c_{i+\mathbf{e}_x}^\dagger - c_{i+\mathbf{e}_y}^\dagger) \qquad \sigma^y c_i^\dagger = \frac{a}{2} \sum_i \hat{D}_i^a,$$

$$\hat{D}_i^a = \sum_\nu \rho_\nu \hat{S}_{i,\nu}^a,$$

where $\rho_{\pm x} = -\rho_{\pm y} = 1$, and the quasilocal d-wave pair $\hat{D}_i^a$ may be seen as the "d-wave unit cell operators." Note the freedom of choosing $a = \pm$ in writing the state. This form makes the physical picture of a d-wave superfluid as delocalized antiferromagnetic order away from half-filling [2] particularly apparent. The condition $[J_i^\alpha, \sum_j \hat{D}_j^b] = 0$ [$\alpha = (a, z)$] is fulfilled by

$$J_i^a = \sum_\nu \rho_\nu j_{i,\nu}^a, \qquad J_i^z = \sum_\nu \rho_\nu j_{i,\nu}^z,$$

with $j_{i,\nu}^z = c_{i+\mathbf{e}_\nu}^\dagger \sigma^z c_i$, establishing Eq. (1). Similar to above, each J_i^a is obtained from $\hat{D}_i^a$ by a particle-hole transformation on the central site i. In fact, for these operators the stronger quasilocal commutation properties with the molecular d-wave pairs hold due to Eq. (3): $[J_i^a, \hat{D}_j^a] = 0$ for all i, j, $[J_i^a, \hat{D}_j^b] = 0$ for all i, j in the same sublattice, which relies again on fermionic statistics. In contrast, the operators J_i^z only commute with the symmetric superposition of all d-wave pairs $\hat{D}_j^a$. These operators establish coherence via phase locking between adjacent cloverleaves of sites.

The dark state uniqueness for the Lindblad operators (1) is equivalent to the uniqueness of the ground state of the associated Hermitian Hamiltonian $H = V \sum_{i,\alpha=\pm,z} J_i^{\alpha\dagger} J_i^\alpha$ for $V > 0$. We note that our BCS state shares the symmetries of H of global phase and spin rotations, and translation invariance. Based on the reasonable assumption that no other symmetries exist, we then expect the ground state to be unique. Note, however, the necessity of the full set $\{J_i^\alpha\}$: Omitting, e.g., $\{J_i^z\}$ gives rise to an additional discrete symmetry of H resulting in ground state degeneracy. These results are confirmed with numerical simulations for small systems (see Fig. 2).

The above construction method allows us to find parent Lindblad operators for a much wider class of BCS-type states. For example, for a $p_x + i p_y$-wave state of spinless fermions, generated by $p^\dagger \sim \sum_{i,\nu} \rho_\nu c_{i+\mathbf{e}_\nu}^\dagger c_i^\dagger$ with $\rho_x = -\rho_{-x} = -i\rho_y = i\rho_{-y} = 1$, the Lindblad operators are $J_i = \sum_\nu \rho_\nu c_{i+\mathbf{e}_\nu}^\dagger c_i$. More generally, they can be obtained for any fixed number pairing state with bilocal pairing [11]. Note, however, that the construction is not applicable for the on site (singlet) pairing states—the analogs of Eq. (1) become local, such that the lattice sites decouple and no phase coherence can be built up.

Physical implementation.—The quasilocal and number-conserving form of J_i^α raises the possibility to realize dissipative pairing via reservoir engineering with cold atoms. We illustrate this, considering alkaline earthlike atoms [12] with nuclear spin (e.g., $I = 1/2$ for ^{171}Yb), and a metastable $^3P_0$ manifold which can be trapped independently to the ground $^1S_0$ manifold. In this setting, one can construct a stroboscopic implementation, where the action of each J_i^α is realized successively. For clarity, we present this initially in 1D, and choose the example of $J_i^+ = (c_{i+1,\uparrow}^\dagger + c_{i-1,\uparrow}^\dagger) c_\downarrow$. The implementation is depicted in Fig. 3: (i) The $^3P_0$ state is trapped in a lattice of 3 times the period as that for the $^1S_0$ state, defining blocks of three sites in the $^1S_0$ lattice. Using this, any $\downarrow$ atom in $^1S_0$ on the central site is excited to the $\uparrow$ state of the $^3P_0$ manifold. (ii) By adding an additional potential the traps for $^3P_0$ are divided so that atoms confined in them overlap the right and left sites of the 3-site block for $^1S_0$. (iii) Dissipation is
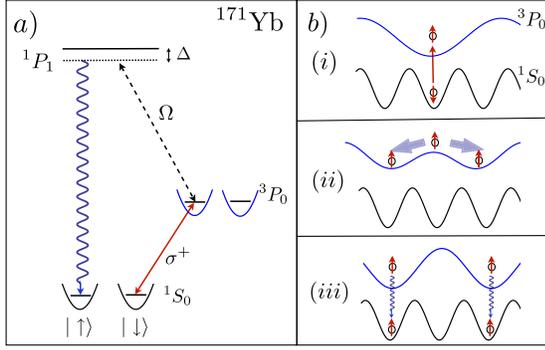
FIG. 3 (color online). (a) Level scheme for alkaline earth atoms with $I = 1/2$, showing excitation to a metastable level while flipping the nuclear spin, and induced decay by coupling to the $^1P_1$ level. (b) Illustration of J_i^+ implementation in 1D: (i) A longer period lattice for $^3P_0$ identifies a triple of wells, and atoms from the central level are transferred to the $^3P_0$ manifold with spin flip. (ii) The $^3P_0$ potential wells are adiabatically split into two; (iii) Decay is induced, returning the atom to the $^1S_0$ level via coupling to a lossy cavity mode.

induced via spontaneous decay, obtained by coupling atoms in the $^3P_0$ state off-resonantly to the $^1P_1$ state, as depicted in Fig. 3(a), with coupling strength Ω, and detuning Δ. If we couple the $^1S_0$-$^1P_1$ transition to a cavity mode with linewidth Γ and vacuum Rabi frequency g, then the decay will be coherent over the triple of sites. In the limit $\Delta \gg \Omega$ and $\Gamma \gg \frac{\Omega g}{\Delta}$, we obtain an effective decay rate $\Gamma_{\text{eff}} = \frac{\Omega^2 g^2}{\Delta^2 \Gamma} \sim 9$ kHz for typical parameters, which bounds the effective dissipative rate for the stroboscopic process, κ. Provided atoms remain in the lowest band, Fermi statistics will be respected, and coherent dynamics can be neglected in a deep lattice for small scattering lengths.

This operation can occur in parallel for different 3-site blocks, and should be repeated with the superlattice shifted for other central sites. Similar operations combined with rotations of the nuclear spin before and after these operations allows implementation of J_i^- and J_i^z. In 2D, 3×3 plaquettes are defined by the appropriate superlattice potential for the $^3P_0$ level, and the adiabatic manipulation of the potential in step (ii) should be adjusted to ensure that the correct relative phases are obtained for atoms transported in orthogonal directions.

The d-wave parent Hamiltonian.—As a final remark, we note that the effective Hamiltonian above can be generalized to include a coherent interaction V,

$$H_{\text{eff}} = (V - \tfrac{i}{2}\kappa)\sum_{i,\alpha} J_i^{\alpha\dagger} J_i^{\alpha}. \tag{4}$$

For $\kappa \to 0$ and interaction $V > 0$ this Hamiltonian can be identified as a *parent Hamiltonian* [13] with $|\text{BCS}_N\rangle$ as unique stable ground state and gapped positive definite excitation spectrum. This parent Hamiltonian could be realized via a similar procedure to the induced dissipation, replacing the decay in step (iii) by induced interactions between atoms. This opens the possibility to use the

d-wave state as an initial state for the preparation of the ground state of the Fermi-Hubbard model by a suitable adiabatic passage [14]. Here, one can take advantage of the fact that (i) in the initial stages the system is protected by a gap ~ 0.72 V, and (ii) the d-wave state has identical symmetry and similar energy to the conjectured Fermi-Hubbard ground state away from half-filling. Thus, since no phase transition has to be crossed, a d-wave superfluid gap protection persists through the whole passage path.

*wyiz@ustc.edu.cn
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