Four-Body Efimov Effect for Three Fermions and a Lighter Particle

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We study three same-spin-state fermions of mass M interacting with a distinguishable particle of mass m in the unitary limit where the interaction has a zero range and an infinite s-wave scattering length. We predict an interval of mass ratio $13.384 < M/m < 13.607$ where there exists a purely four-body Efimov effect, leading to the occurrence of weakly bound tetramers without Efimov trimers.

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In a system of interacting particles, the unitary limit corresponds to a zero-range s-wave interaction with infinite scattering length [1]. In particular, this excludes any finite energy two-body bound state. Interestingly, in the three-body problem, the Efimov effect may take place [2], leading to the occurrence of an infinite number of three-body bound states, with an accumulation point in the spectrum at zero energy. This effect occurs in a variety of situations, the historical one being the case of three bosons, as recently studied in a series of remarkable experiments with cold atoms close to a Feshbach resonance [3]. It also occurs in the $2+1$ fermionic problem of two same-spin-state fermions of mass M interacting only with a particle of another species of mass m, for a mass ratio $\alpha = M/m$ larger than $\alpha_c(2;1) \simeq 13.607$ [2].

The four-body problem with large scattering length has recently attracted a lot of interest [4]. In this resonant regime, the Efimov effect for four bosons was pointed out in Ref. [5] to be washed out by the presence of Efimov trimers. An alternative proposed in Ref. [5] and further explored in Refs. [6,7] was to leave the unitary limit and consider a three-body resonant regime. In this Letter, we stick to the unitary limit and show that the $3+1$ fermionic problem, unlike the $3+1$ bosonic one [7], exhibits a four-body Efimov effect, within an interval of mass ratio where Efimov trimers are absent. This is obtained by explicitly solving Schrödinger's equation in the zero-range model [2] thanks to the scaling invariance of the model [8].

In the zero-range model, the Hamiltonian reduces to a noninteracting form, here in free space

$$H = \sum_{i=1}^{4} -\frac{\hbar^2}{2m_i}\Delta_{\mathbf{r}_i}, \qquad (1)$$

with $m_1 = m_2 = m_3 = M$ and $m_4 = m$. The interactions are indeed replaced by contact conditions on the wave function, $\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$, where $\mathbf{r}_i$, $i = 1, 2, 3$, is the position of a fermion and $\mathbf{r}_4$ is the position of the other species particle: At the unitary limit, for $i = 1, 2, 3$, there exist functions A_i such that

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \frac{A_i(\mathbf{R}_{i4}; (\mathbf{r}_k)_{k\neq i,4})}{|\mathbf{r}_i - \mathbf{r}_4|} + O(|\mathbf{r}_i - \mathbf{r}_4|) \qquad (2)$$

when $\mathbf{r}_i$ tends to $\mathbf{r}_4$ for a fixed value of the $(i, 4)$ centroid $\mathbf{R}_{i4} \equiv (M\mathbf{r}_i + m\mathbf{r}_4)/(m + M)$ different from the positions of the remaining particles $\mathbf{r}_k$, $k \neq i, 4$. The wave function is also subject to the fermionic exchange symmetry with respect to the first three variables $\mathbf{r}_i$, $i = 1, 2, 3$.

In what follows, we shall assume that there is no three-body Efimov effect, a condition that is satisfied by imposing $M/m < \alpha_c(2; 1) \simeq 13.607$. The eigenvalue problem $H\psi = E\psi$ with the contact conditions in Eq. (2) is then separable in hyperspherical coordinates [8]. After having separated out the center of mass $\mathbf{C}$ of the system, one introduces the hyperradius $R = [\sum_{i=1}^4 m_i(\mathbf{r}_i - \mathbf{C})^2/\bar{m}]^{1/2}$, with $\bar{m} = (3M + m)/4$ the average mass, and a set of here 8 hyperangles Ω whose expression is not required. For a center of mass at rest, the wave function may be taken of the form

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = R^{-7/2}F(R)f(\Omega). \qquad (3)$$

$f(\Omega)$ is given by the solution of a Laplacian eigenvalue problem on the unit sphere of dimension eight, which is nontrivial because of the contact conditions. On the contrary, the hyperradial part F is not directly affected by the contact conditions, due, in particular, to their invariance by the scaling $\mathbf{r}_i \to \lambda\mathbf{r}_i$ [9], and solves the effective 2D Schrödinger equation

$$EF(R) = -\frac{\hbar^2}{2\bar{m}}\left(\partial_R^2 + \frac{1}{R}\partial_R\right)F(R) + \frac{\hbar^2 s^2}{2\bar{m}R^2}F(R). \qquad (4)$$

The quantity s^2 is given by the hyperangular eigenvalue problem. This problem is self-adjoint, and thus s^2 belongs to an infinite discrete set and is real, since there is no Efimov effect on the unit sphere ($R \neq 0$), that is, here no three-body Efimov effect.

Mathematically, Eq. (4) admits for all energies E two linearly independent solutions, respectively, behaving as $R^{\pm s}$ for $R \to 0$. If $s^2 > 0$, one imposes $F(R) \sim R^s$, with $s > 0$, which is correct except for accidental, nonuniversal four-body resonances (see [10] and note [43] in [8]) recently found numerically [11]. Equation (4) then does not support any bound state. On the contrary, if $s^2 < 0$, in which case we set $s = iS$, $S > 0$, F experiences an effective four-body attraction, with a fall to the center leading to an unphysical continuous spectrum of bound states [12]. To make the model self-adjoint, one then imposes an extra contact condition [12], as in the usual three-body Efimov case [13]:

$$F(R) \underset{R\to 0}{\sim} \mathrm{Im}\left[\left(\frac{R}{R_f}\right)^{iS}\right], \tag{5}$$

where the four-body parameter R_f depends on the microscopic details of the true, finite-range interaction [14]. With the extra condition Eq. (5), one then obtains from Eq. (4) an Efimov spectrum of tetramers:

$$E_n = -\frac{2\hbar^2}{\bar{m}R_f^2} e^{(2/S)\arg\Gamma(1+iS)} e^{-2\pi n/S} \quad \forall\ n \in \mathbb{Z}. \tag{6}$$

The whole issue is thus to determine the possible values of the exponents s. In particular, the critical mass ratio $\alpha_c(3;1)$ corresponds to one of the exponents being equal to zero, the other ones remaining positive. To this end, we calculate the zero energy four-body wave function with no specific boundary condition on $F(R)$. Then, from Eq. (4) with $E = 0$, it appears that $F(R) \propto R^{\pm s}$. The calculation is done in momentum space, with the ansatz for the Fourier transform of the four-body wave function:

$$\tilde{\psi}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \frac{\delta(\sum_{i=1}^4 \mathbf{k}_i)}{\sum_{i=1}^4 \frac{\hbar^2 k_i^2}{2m_i}}[D(\mathbf{k}_2, \mathbf{k}_3) + D(\mathbf{k}_3, \mathbf{k}_1) + D(\mathbf{k}_1, \mathbf{k}_2)], \tag{7}$$

where the fermionic symmetry imposes $D(\mathbf{k}_2, \mathbf{k}_1) = -D(\mathbf{k}_1, \mathbf{k}_2)$ and the denominator originates from the action of H in Eq. (1) written in momentum space. When H acts on one of the three $1/|\mathbf{r}_4 - \mathbf{r}_i|$ singularities in Eq. (2), this produces in the right-hand side of Schrödinger's equation a Dirac distribution $\delta(\mathbf{r}_4 - \mathbf{r}_i)$ multiplied by a translationally invariant function of the three fermionic positions, which after Fourier transform gives each of the D terms inside the square brackets of Eq. (7). Taking the Fourier transform of Eq. (3) with $F(R) \propto R^{\pm s}$, and using a power-counting argument, one finds the scaling law

$$D(\lambda\mathbf{k}_1, \lambda\mathbf{k}_2) = \lambda^{-(\pm s + 7/2)} D(\mathbf{k}_1, \mathbf{k}_2). \tag{8}$$

Implementing in momentum space the contact conditions, that is, the fact that $O(|\mathbf{r}_i - \mathbf{r}_4|)$ vanishes for $\mathbf{r}_i = \mathbf{r}_4$ in Eq. (2), gives rise to an integral equation [16]:

$$\left[\frac{1+2\alpha}{(1+\alpha)^2}(k_1^2 + k_2^2) + \frac{2\alpha}{(1+\alpha)^2}\mathbf{k}_1\cdot\mathbf{k}_2\right]^{1/2} D(\mathbf{k}_1, \mathbf{k}_2) = \int \frac{d^3k_3}{2\pi^2} \frac{-[D(\mathbf{k}_1, \mathbf{k}_3) + D(\mathbf{k}_3, \mathbf{k}_2)]}{k_1^2 + k_2^2 + k_3^2 + \frac{2\alpha}{1+\alpha}(\mathbf{k}_1\cdot\mathbf{k}_2 + \mathbf{k}_1\cdot\mathbf{k}_3 + \mathbf{k}_2\cdot\mathbf{k}_3)}, \tag{9}$$

where we recall that $\alpha = M/m$. Equation (9) can also be obtained as the zero-range limit of finite-range models [17].

We now use rotational invariance to impose the value $l \in \mathbb{N}$ of the total angular momentum of the four-body state and to restrict to a zero angular momentum along the quantization axis z. Then, according to Eq. (7), the effective two-body function $D(\mathbf{k}_1, \mathbf{k}_2)$ has the same angular momentum l. This allows us to express D in terms of $2l + 1$ unknown functions $f_{m_l}^{(l)}$ of three real variables only, the moduli k_1 and k_2 and the angle $\theta \in [0, \pi]$ between $\mathbf{k}_1$ and $\mathbf{k}_2$, with the fermionic symmetry imposing $f_{m_l}^{(l)}(k_2, k_1, \theta) = (-1)^{l+1} f_{-m_l}^{(l)}(k_1, k_2, \theta)$ [17]:

$$D(\mathbf{k}_1, \mathbf{k}_2) = \sum_{m_l=-l}^{l} [Y_l^{m_l}(\gamma, \delta)]^* e^{i m_l \theta/2} f_{m_l}^{(l)}(k_1, k_2, \theta). \tag{10}$$

Here $Y_l^{m_l}(\gamma, \delta)$ are the usual spherical harmonics, and γ and δ are the polar and azimuthal angles, respectively, of the unit vector $\mathbf{e}_z$ along z in the direct orthonormal basis $(\mathbf{e}_1, \mathbf{e}_{2\perp}, \mathbf{e}_{12})$, with $\mathbf{e}_1 = \mathbf{k}_1/k_1$, $\mathbf{e}_2 = \mathbf{k}_2/k_2$, $\mathbf{e}_{2\perp} = (\mathbf{e}_2 - \mathbf{e}_1 \cos\theta)/\sin\theta$, and $\mathbf{e}_{12} = \mathbf{e}_1 \times \mathbf{e}_2/\sin\theta$ [18]. The action of parity $\mathbf{k}_i \to -\mathbf{k}_i$ on this general ansatz is to multiply each term of index m_l in Eq. (10) by a factor $(-1)^{m_l}$, which allows decoupling of the even m_l terms (even parity) from the odd m_l terms (odd parity). A relevant example, as we shall see, is the even parity channel with $l = 1$, where the ansatz reduces to a single term, which is obviously the component along z of a vectorial spinor:

$$D(\mathbf{k}_1, \mathbf{k}_2) \propto \mathbf{e}_z \cdot \frac{\mathbf{k}_1 \times \mathbf{k}_2}{\|\mathbf{k}_1 \times \mathbf{k}_2\|} f_0^{(1)}(k_1, k_2, \theta). \tag{11}$$

The last step is to use the scaling invariance of D [see Eq. (8)], setting

$$f_{m_l}^{(l)}(k_1, k_2, \theta) = (k_1^2 + k_2^2)^{-(s+7/2)/2}(\cosh x)^{3/2}\Phi_{m_l}^{(l)}(x, u), \tag{12}$$

where $u = \cos\theta$. The introduction of the logarithmic change of variable $x = \ln(k_2/k_1)$ is motivated by Efimov physics, and the factor involving the hyperbolic cosine ensures that the final integral equation involves a Hermitian operator. The fermionic symmetry imposes

$$\Phi_{m_l}^{(l)}(-x, u) = (-1)^{l+1}\Phi_{-m_l}^{(l)}(x, u), \tag{13}$$

which allows us to restrict the unknown functions $\Phi_{m_l}^{(l)}$ to $x \geq 0$. Restricting to $s = iS$, $S \geq 0$, we finally obtain

$$0 = \left[\frac{1+2\alpha}{(1+\alpha)^2} + \frac{\alpha u}{(1+\alpha)^2 \cosh x}\right]^{1/2} \Phi_{m_l}^{(l)}(x, u) + \int_{\mathbb{R}^+} dx' \int_{-1}^{1} du' \sum_{m_l'=-l}^{l} \mathcal{K}_{m_l, m_l'}^{(l)}(x, u; x', u') \Phi_{m_l'}^{(l)}(x', u'). \tag{14}$$

The symmetrized kernel $\mathcal{K}_{m_l, m_l'}^{(l)}(x, u; x', u') = \sum_{\epsilon, \epsilon'=\pm 1} (\epsilon\epsilon')^{l+1} K_{\epsilon m_l, \epsilon' m_l'}^{(l)}(\epsilon x, u; \epsilon' x', u')$ is expressed in terms of the nonsymmetrized one given by

$$K_{m_l,m_l'}^{(l)}(x,u;x',u') = \frac{[(1+\lambda^2)/(1+\lambda'^2)]^{iS/2}(\lambda\lambda')^{3/2}}{[(1+\lambda^2)(1+\lambda'^2)]^{1/4}} \int_0^{2\pi} \frac{d\phi}{2\pi^2} \frac{e^{-im_l\theta/2}\langle l, m_l|e^{i\phi L_x/\hbar}|l, m_l'\rangle e^{im_l'\theta'/2}}{1+\lambda^2+\lambda'^2+\frac{2\alpha}{1+\alpha}[\lambda u+\lambda' u'+\lambda\lambda'\mathcal{D}]}. \quad (15)$$

Here the notation $\mathcal{D}$ in the denominator stands for $\mathcal{D} = uu' + \cos\phi\sqrt{1-u^2}\sqrt{1-u'^2}$, $\lambda = e^x$, $\lambda' = e^{x'}$, L_x is the angular momentum operator along x, $|l, m_l\rangle$ is of spin l and angular momentum $m_l\hbar$ along z, and ϕ stands for the azimuthal angle of the vector $\mathbf{k}_3$ of Eq. (9) in the spherical coordinates related to the basis $(\mathbf{e}_{2\perp}, \mathbf{e}_{12}, \mathbf{e}_1)$ [19].

We first look for the critical mass ratio for the 3 + 1 fermionic problem $\alpha_c(3;1)$, which is the minimal value of α such that the integral equation (14) is satisfied for $S = 0$. Rewriting Eq. (14) as $0 = M_s[\Phi]$, where M_s is a Hermitian operator, we calculated numerically the minimal eigenvalues of $M_{s=0}$ as functions of the mass ratio α, within each subspace of fixed parity and angular momentum l, $0 \le l \le 6$. As shown in Fig. 1, such a minimal eigenvalue vanishes for $\alpha < 13.607$ only in the even sector of angular momentum $l = 1$. We also unfruitfully explored $l = 7, 8, 9, 10$. We thus find that the four-body Efimov effect takes place only in the even sector of $l = 1$ and sets in above a critical mass ratio [20]

$$\alpha_c(3;1) \simeq 13.384, \quad (16)$$

quite close to the 2 + 1 critical value $\alpha_c(2;1) \simeq 13.607$.

To gain some insight on this result, we have studied analytically an important feature of the spectrum of $M_{s=0}$, the lower border of its continuum. When $x, x' \to +\infty$, which corresponds to having $k_2 \gg k_1$ in the function $D(\mathbf{k}_1, \mathbf{k}_2)$, both the symmetrized and nonsymmetrized kernels reduce to the asymptotic form

$$K_{m_l,m_l'}^{(l)}(x,u;x',u') \sim e^{iS(x-x')}e^{-im_l\theta/2}e^{im_l'\theta'/2}\int_0^{2\pi}\frac{d\phi}{4\pi^2} \times \frac{\langle l, m_l|e^{i\phi L_x/\hbar}|l, m_l'\rangle}{\cosh(x-x')+\frac{\alpha}{1+\alpha}\mathcal{D}}. \quad (17)$$

Since $\mathcal{D}$ is independent of x and x', this is invariant by translation over the x coordinates, leading to a continuous spectrum of asymptotic plane wave eigenfunctions. In the even sector of angular momentum $l = 1$, we found that $\Phi_0^{(1)}(x,u) \sim e^{ikx}\sqrt{1-u^2}$ gives rise to an eigenfunction in the continuous spectrum of M_{iS} with the real eigenvalue $\Lambda(k - S, \alpha)$ [21] where

$$\Lambda(k,\alpha) = \cos 2\beta + \frac{(1-ik)\sin[2\beta(1+ik)] - \text{c.c.}}{2(1+k^2)\sin^2 2\beta \sin(ik\pi/2)}. \quad (18)$$

In Eq. (18) we have set for convenience $\sin 2\beta = \alpha/(1+\alpha)$ with $\beta \in [0, \pi/4]$. For real k, this function $\Lambda(k,\alpha)$ has a global minimum in $k = 0$. We expect that $\Lambda(k = 0, \alpha)$ is the lower border of the continuous spectrum of $M_{s=0}$. Since $\Lambda(0,\alpha)$ exactly vanishes for the three-body critical mass ratio $\alpha_c(2;1) \simeq 13.607$, our asymptotic analysis amounts to uncovering the three-body problem as a limit $k_2/k_1 \to +\infty$ of the four-body problem [22].

We tested this prediction against the numerics, plotting in Fig. 1 the quantity $\Lambda(k = 0, \alpha)$ as a function of α (dotted line). Except for the even sector of $l = 1$, the minimal numerical eigenvalues are close to $\Lambda(k = 0, \alpha)$; the fact that they are slightly above is due to a finite $x_{\max}$ truncation effect that indeed decreases for increasing $x_{\max}$ (not shown). This implies that the eigenfunctions corresponding to these minimal eigenvalues are extended, that is, not square integrable. The numerics agrees with this analysis. In the even sector of $l = 1$, the minimal numerical eigenvalue is clearly below $\Lambda(0,\alpha)$, for all values of α in Fig. 1. This indicates that the corresponding eigenvector must be a bound state of $M_{s=0}$, with a square integrable eigenfunction $\Phi_0^{(1)}(x,u)$. This is confirmed by the numerics, which shows that at large x, $\Phi_0^{(1)}(x,u) \propto \sqrt{1-u^2}e^{-\kappa x}$. The analytical reasoning even predicts the link between the minimal eigenvalue $\Lambda_{\min}$ of $M_{s=0}$ and the decay constant κ: The plane wave e^{ikx} is analytically continuated into a decreasing exponential if one sets $k = i\kappa$, so that $\Lambda_{\min} = \Lambda(i\kappa, \alpha)$ (this also holds for $S > 0$). Numerically, we have successfully tested this relation for various values of α, and we also found that $M_{s=0}$ has no other bound state in the even sector of $l = 1$.

Finally, we completed our study of the four-body Efimov effect by calculating, as a function of the mass ratio α, the exponent $s = iS$ in the even sector of $l = 1$, the real quantity S being such that the operator M_{iS} has a zero eigenvalue. The result is shown in Fig. 2. Close to the 2 + 1

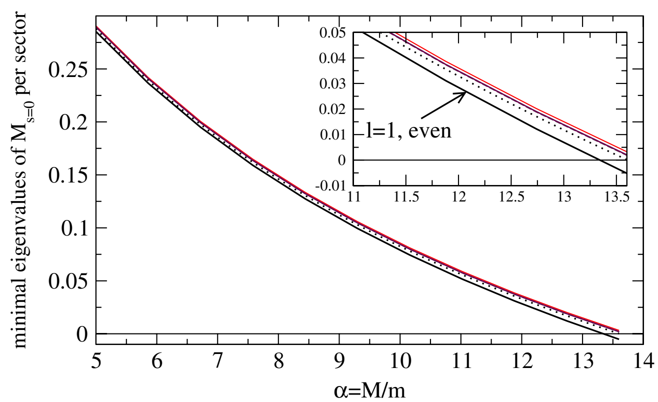

FIG. 1 (color online). Minimal eigenvalues of the Hermitian operators $M_{s=0}$ in each sector of fixed parity and angular momentum l, $0 \le l \le 6$, as functions of the mass ratio $\alpha = M/m$. Only the curve for the even sector of $l = 1$ crosses zero for $\alpha < 13.607$, corresponding to the occurrence of a four-body Efimov effect in that sector. The other curves all remain above zero. They strongly overlap and are barely distinguishable at the scale of the figure. The dotted line is the analytical prediction $\Lambda(k = 0, \alpha)$ for the lower border of the continuum in the spectrum of $M_{s=0}$. The inset is a magnification. In the numerics, x and u were discretized with a step $dx = du = 1/10$, and x was truncated to $x_{\max} = 20$.

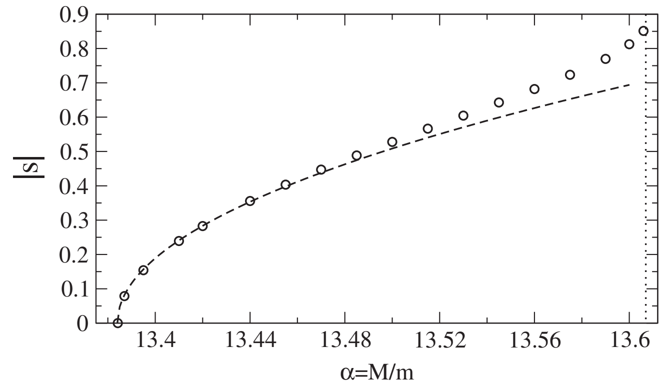

FIG. 2. In the Efimovian channel $l = 1$ with even parity, modulus of the purely imaginary Efimov exponent $s = iS$ as a function of the mass ratio $\alpha = M/m$. In the numerics, $x_{\max}$ ranges from 40 to 120, $dx = 1/10$, $d\theta = \pi/20$. The dashed line results from a linear fit of $|s|^2$ as a function of α in a vicinity of the critical value $\alpha_c(3;1)$, $|s|^2_{\rm fit} \simeq 2.23(\alpha - \alpha_c)$. The vertical dotted line indicates the 2 + 1 critical value $\alpha_c(2;1)$.

critical mass ratio $\alpha_c(2;1) \simeq 13.607$, the values of $|s|$ are of order unity, which shall give a sizable experimental effect [3]. Close to the 3 + 1 critical mass ratio $\alpha_c(3;1)$, $|s|$ varies as $(\alpha - \alpha_c)^{1/2}$ (see the dashed line) [23]. Low values of $|s|$ lead to extremely low Efimov tetramer energies: For an interaction of finite-range b, setting $R_f \approx b$ and $n = 1$ in Eq. (6), we estimate the ground state Efimov tetramer energy for $|s| \ll 1$ as $E_{\min}^{\rm Efim} \approx -e^{-2\pi/|s|}\hbar^2/(2\bar{m}b^2)$ [24]. For $|s| = 0.5$, taking the mass of ^{3}He for m and a few nanometers for b gives $E_{\min}^{\rm Efim}/k_B$ in the nanokelvin range, accessible to cold atoms. Moreover, for a large but finite scattering length a, successive Efimov tetramers come in for values of a in geometric progression of ratio $e^{\pi/|s|}$, so that too low values of $|s|$ require unrealistically large values of a. Another experimental issue is the narrowness of the mass interval. Several pairs of atomic species have a mass ratio in the desired interval, e.g., ^{3}He* and ^{41}Ca ($\alpha \simeq 13.58$), and with exotic species, ^{11}B and ^{149}Sm ($\alpha \simeq 13.53$) and ^{7}Li and ^{95}Mo ($\alpha \simeq 13.53$). A more flexible solution is to start with usual atomic species having a slightly off-mass ratio, such as ^{3}He* and ^{40}K ($\alpha \simeq 13.25$), and to use a weak optical lattice to finely tune the effective mass of one of the species [25].

In conclusion, in the zero-range model at unitarity, we studied the interaction of three same-spin-state fermions of mass M with another particle of mass m. For $M/m < 13.384$, no Efimov effect was found. Over the interval $13.384 < M/m < 13.607$, remarkably a purely four-body Efimov effect takes place, in the sector of even parity and angular momentum $l = 1$, that may be observed with a dedicated cold atom experiment. For $M/m > 13.607$, the three-body Efimov effect sets in, and the zero-range model has to be supplemented by three-body contact conditions that break its separability. The intriguing question of whether the Efimov tetramers then survive as resonances, decaying in a trimer plus a free atom, is left for the future.

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[1] E. Braaten and H.-W. Hammer, Phys. Rep. **428**, 259 (2006).
[2] V. Efimov, Sov. J. Nucl. Phys. **12**, 589 (1971); Nucl. Phys. **A210**, 157 (1973); A. Bulgac and V. Efimov, Sov. J. Nucl. Phys. **22**, 296 (1975).
[3] T. Kraemer *et al.*, Nature (London) **440**, 315 (2006); M. Zaccanti *et al.*, Nature Phys. **5**, 586 (2009); N. Gross *et al.*, Phys. Rev. Lett. **103**, 163202 (2009); **105**, 103203 (2010); S. E. Pollack, D. Dries, and R. G. Hulet, Science **326**, 1683 (2009); T. Lompe *et al.*, Science **330**, 940 (2010).
[4] D. S. Petrov, C. Salomon, and G. Shlyapnikov, Phys. Rev. Lett. **93**, 090404 (2004); C. Mora *et al.*, Phys. Rev. Lett. **95**, 080403 (2005); H.-W. Hammer and L. Platter, Eur. Phys. J. A **32**, 113 (2007); J. von Stecher, J. P. D'Incao, and C. H. Greene, Nature Phys. **5**, 417 (2009); K. M. Daily and D. Blume, Phys. Rev. A **81**, 053615 (2010).
[5] R. Amado and F. Greenwood, Phys. Rev. D **7**, 2517 (1973).
[6] H. W. L. Naus and J. A. Tjon, Few-Body Syst. **2**, 121 (1987); H. Kröger and R. Perne, Phys. Rev. C **22**, 21 (1980).
[7] S. Adhikari and A. Fonseca, Phys. Rev. D **24**, 416 (1981).
[8] F. Werner and Y. Castin, Phys. Rev. A **74**, 053604 (2006).
[9] Another point is that R varies to second order in $|\mathbf{r}_i - \mathbf{r}_4|$ when $\mathbf{r}_i$ tends to $\mathbf{r}_4$ for a fixed $\mathbf{R}_{i4}$; cf. note [38] in [8].
[10] Y. Nishida, D. T. Son, and S. Tan, Phys. Rev. Lett. **100**, 090405 (2008).
[11] S. Gandolfi and J. Carlson, arXiv:1006.5186; D. Blume and K. M. Daily, Phys. Rev. Lett. **105**, 170403 (2010).
[12] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. II, p. 1665.
[13] G. S. Danilov, Sov. Phys. JETP **13**, 349 (1961).
[14] For a narrow resonance, R_f may be proportional to the effective range as for the three-body Efimov effect [15].
[15] D. S. Petrov, Phys. Rev. Lett. **93**, 143201 (2004); A. Gogolin, C. Mora, and R. Egger, Phys. Rev. Lett. **100**, 140404 (2008).
[16] L. Pricoupenko, arXiv:1007.4939.
[17] C. Mora, Y. Castin, and L. Pricoupenko (to be published).
[18] One thus has $\cos\gamma = \mathbf{e}_z \cdot \mathbf{e}_{12}$, $\sin\gamma\cos\delta = \mathbf{e}_z \cdot \mathbf{e}_1$, and $\sin\gamma\sin\delta = \mathbf{e}_z \cdot \mathbf{e}_{2\perp}$, with $\gamma \in [0,\pi]$ and $\delta \in [0, 2\pi]$.
[19] The matrix elements $\langle l, m_l|e^{i\phi L_x/\hbar}|l, m_l'\rangle$ are evaluated by insertion of a closure relation in the eigenbasis of L_x. One then faces integrals $J_n = \int_0^{2\pi} d\phi \frac{e^{in\phi}}{b_0 + b_1\cos\phi} = 2\pi z_0^{|n|}/\sqrt{b_0^2 - b_1^2}$ with $z_0 = -(b_0/b_1) + \sqrt{(b_0/b_1)^2 - 1}$.
[20] To gain in precision, we used (x, θ) as variables ($x_{\max}$ up to 40 and $dx = d\theta/\pi$ down to 1/40) and $\sqrt{\sin\theta}\Phi_{m_l}^{(l)}(x, \cos\theta)$ as the unknown function to preserve Hermiticity.
[21] This is more rapidly obtained by taking $k_2 \to \infty$ in Eq. (9) with the ansatz $D(\mathbf{k}_1, \mathbf{k}_2) \sim \mathbf{e}_z \cdot \mathbf{e}_1 \times \mathbf{e}_2 k_2^{-7/2}(k_2/k_1)^{ik+3/2}$.
[22] $\Lambda(k, \alpha)$ can indeed be related to the function $\lambda(\nu = ik - 1)$ of D. Petrov, Phys. Rev. A **67**, 010703 (2003).
[23] The lowest eigenvalue of M_{iS} departs from zero linearly in $\alpha - \alpha_c(3;1)$ and quadratically in S.
[24] Imposing $F(R = b) = 0$ also gives this estimate. Cutting s^2/R^2 to s^2/b^2 for $R < b$, as in L. D. Landau and L. M. Lifshitz, *Quantum Mechanics* (Butterworth-Heinemann, London, 1981), gives $E_{\min}^{\rm Efim} \approx -e^{-\pi/|s|}2\hbar^2/(\bar{m}b^2)$.
[25] D. S. Petrov *et al.*, Phys. Rev. Lett. **99**, 130407 (2007).