

Existence of Majorana Fermions and Topological Order in Nodal Superconductors with Spin-Orbit Interactions in External Magnetic Fields

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(Received 26 July 2010; published 15 November 2010)

We demonstrate that Majorana fermions exist in edges of systems and in a vortex core even for superconductors with nodal excitations such as the d -wave pairing state under a particular but realistic condition in the case with an antisymmetric spin-orbit interaction and a nonzero magnetic field below the upper critical field. We clarify that the Majorana fermion state is topologically protected in spite of the presence of bulk gapless nodal excitations, because of the existence of a nontrivial topological number. Our finding drastically enlarges target systems where we can explore the Majorana fermion state.

DOI: 10.1103/PhysRevLett.105.217001

PACS numbers: 74.72.-h, 71.10.Pm, 74.25.Ha, 74.90.+n

Introduction.—Majorana fermions (MFs) realized as vortex core bound states of superconducting condensates have been attracting considerable interest in connection with the application to quantum computation [1–3]. Such vortices obey the non-Abelian statistics [4–9], and because of this distinct feature, they are utilized as decoherence-free qubits. The realization of Majorana bound states has been discussed for the quantum Hall effect systems [4–7], $p + ip$ superconductors [7–11], superconductor-topological-insulator interfaces [12,13], and s -wave Rashba superconductors [14–16]. The origin of MFs acting as non-Abelian anyons is intimately related to the existence of the non-Abelian topological order, which yields the fractionalization of quasiparticles [17,18]. Generally, topological order is characterized by a nontrivial topological number associated with the global structure of the Hilbert space and, hence, the existence of a nonzero energy gap which separates the topological ground state and nontopological excited states stabilizes the topological order.

In this Letter, we propose an example of systems realizing MFs, which is unusual in the above-mentioned sense of topological stability but frequently found in real materials: MF states realized in superconductors with nodal excitations such as the d -wave pairing state. More precisely, MFs exist in edges of systems and in a vortex core for nodal superconductors, provided that there are both an antisymmetric spin-orbit interaction and a nonzero magnetic field below the upper critical field H_{c2} and that, when the magnetic field is switched off, the Fermi level is located close to odd numbers of time-reversal invariant \mathbf{k} points in the Brillouin zone (BZ) at which the superconducting gap vanishes because of the symmetry requirement. This proposal implies that the non-Abelian topological order coexists with gapless excitations in our system. One may wonder how the topological stability is ensured in the presence of nontopological gapless excitations. In fact, the Chern number is not well-defined in our nodal system. Nevertheless, we clarify that the MF state is, to some

extent, stable against interactions with nodal excitations and with impurities because of the existence of a topological number which is well-defined even for gapless superconductors. Note that many classes of noncentrosymmetric (NCS) superconductors, such as CePt₃Si, CeRhSi₃, CeIrSi₃, and Li₂Pt₃B, are known to possess superconducting gap nodes [19–23]. In these systems, some of time-reversal invariant \mathbf{k} points reside close to the Fermi level [24]. Our finding indicates that if the total number of these \mathbf{k} points is odd, and the superconducting gap vanishes (or, at least, becomes sufficiently small) at these points, stable MF modes appear under applied magnetic fields. We expect that such MF states may be realized in large classes of NCS superconductor with gap nodes.

Majorana fermions in edges and in a vortex core.—To be concrete, we consider a two-dimensional d -wave superconductor with the Rashba spin-orbit interaction, though the following argument is basically applicable to any NCS nodal superconductors. The Hamiltonian is given by $\mathcal{H} = \frac{1}{2} \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} \mathcal{H}(\mathbf{k}) \psi_{\mathbf{k}}$, with

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} \epsilon_{\mathbf{k}} - h\sigma_z + \mathbf{g}_{\mathbf{k}} \cdot \boldsymbol{\sigma} & i\Delta_{\mathbf{k}}\sigma_y \\ -i\Delta_{\mathbf{k}}\sigma_y & -\epsilon_{\mathbf{k}} + h\sigma_z + \mathbf{g}_{\mathbf{k}} \cdot \boldsymbol{\sigma}^* \end{pmatrix}, \quad (1)$$

where $\psi_{\mathbf{k}}^{\dagger} = (c_{k\uparrow}^{\dagger}, c_{k\downarrow}^{\dagger}, c_{-k\uparrow}, c_{-k\downarrow})$, $\epsilon_{\mathbf{k}} = -2t(\cos k_x + \cos k_y) - \mu$, $\mathbf{g}_{\mathbf{k}} = 2\lambda(\sin k_y, -\sin k_x, 0)$, $\mathbf{k} = (k_x, k_y)$, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. $h = \mu_B H_z$ is a Zeeman magnetic field. The gap function is the $d_{x^2-y^2}$ -wave type, $\Delta_{\mathbf{k}} = \Delta_0(\cos k_x - \cos k_y)$, or the d_{xy} -wave type, $\Delta_{\mathbf{k}} = \Delta_0 \sin k_x \sin k_y$. We neglect the orbital effect of the magnetic field for a while since it does not change our results qualitatively, as long as $H_z < H_{c2}$.

We first demonstrate that there is a gapless chiral MF mode on the edge of the system. For this purpose, we numerically calculate the energy spectrum of the system with the open boundary condition imposed for the x axis and the periodic boundary condition for the y axis. The

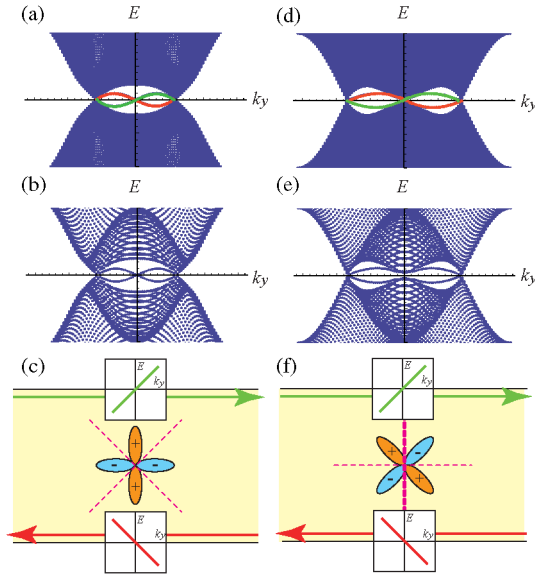


FIG. 1 (color online). Energy spectra for systems with open boundaries for the x direction and the periodic boundary condition for the y direction. $\mu = -4t$, $\lambda = 0.5t$, $\Delta_0 = t$, and $h = 2t$. (a),(b) $d_{x^2-y^2}$ -wave pairing. (d),(e) d_{xy} -wave pairing. The distance between two edges is $L = 90$ [(a) and (d)] and $L = 30$ [(b) and (e)]. In (a) and (d), chiral gapless edge modes at $x = 0$ and $x = L$ are depicted, respectively, in green and red curves. (c),(f) Chiral edge modes counterpropagating on two opposite edges for $d_{x^2-y^2}$ -wave pairing (c) and d_{xy} -wave pairing (f).

results are shown in Fig. 1. In the case of the $d_{x^2-y^2}$ -wave pairing, when the condition $-4t - \mu < h < -\mu$ is satisfied, a gapless edge mode appears for $k_y \sim 0$ [Fig. 1(a)]. Note that this edge mode is isolated from the continuum of gapless excitations from the gap nodes with the finite Fermi momentum. Because of the particle-hole symmetry of the Hamiltonian, the existence of one zero energy mode for $k_y \sim 0$ implies that it is a MF mode. The above condition implies that when the Fermi level crosses \mathbf{k} points close to time-reversal invariant points in the BZ in the absence of the magnetic field, say, the Γ point, i.e., $\mu = -4t$, the Majorana edge mode appears for any nonzero H_z below H_{c2} . This property makes a sharp contrast to the s -wave pairing state considered in Refs. [14–16], for which a large magnetic field satisfying $h > \Delta$ is required to realize the topological order and MFs. This is because the d -wave gap function vanishes at $\mathbf{k} \sim 0$, fulfilling the condition $h > \Delta_{\mathbf{k} \sim 0} \sim 0$. As a result, the realization of MFs in the d -wave pairing state is much more feasible than that in the s -wave pairing state, which may be seriously affected by the orbital depairing effect due to the large magnetic field $h > \Delta$.

In the case of the d_{xy} -wave pairing, the zero energy edge mode at $k_y \sim 0$ merges into nodal excitations, and, thus, it is difficult to identify the Majorana mode from the numerical result [Fig. 1(d)]. However, we deduce that there is still a Majorana mode at $k_y \sim 0$ also for this case, because of the topological argument which will be presented later, as

long as L is sufficiently large. (Also see the argument on the stability below.)

The existence of the chiral MF edge mode implies that there is a MF mode in a vortex core of the superconducting condensate when the vorticity n is odd. In the case of the d -wave pairing, the analysis of the vortex core state is cumbersome, in contrast to the s -wave pairing or the $p + ip$ -wave pairing states, since the gap function of the d -wave state is not an eigenstate of the orbital angular momentum, and, moreover, there is no truly localized bound state in a vortex core because of interactions with delocalized nodal excitations [25,26]. However, as in the case of the s -wave Rashba superconductor [14,27], we can construct the Majorana zero energy mode from quasiparticles with $\mathbf{k} \sim 0$ at least in some parameter regions. The eigenfunction for this zero energy state is $\phi^T = (u_1, u_1, u_1^*, u_1^*)$, with $u_1 = ie^{i[(n-1)/2]\theta} f(r)$, $u_1 = -ie^{i[(n+1)/2]\theta} f(r)$, and $f(r) = \sqrt{h/\pi\lambda r} e^{-(h/2\lambda)r}$ for large r . Here n is odd. In the d -wave pairing state, in addition to this zero energy mode, there are four gapless extended states outside of the vortex core which stem from the four gap nodes [28,29]. Since the total number of the zero energy mode is odd, one Majorana mode survives. Thus, we have a zero energy MF mode in the vortex core.

Stability of Majorana fermions.—The next important question is whether MFs found above are stable or not against weak perturbations such as impurities even in the presence of gapless nodal excitations. As will be shown later, there is indeed a topological protection mechanism in spite of the existence of bulk gapless excitations. Before discussing the topological mechanism, however, we here present a heuristic argument on this issue to grasp an intuitive physical picture. Generally, impurity scattering affects the superconducting state with gap nodes. We consider only the case of weak disorder for which the superconducting gap is not much reduced. We first consider the case of the $d_{x^2-y^2}$ -wave pairing. In a semi-infinite system with an open boundary, there is only one chiral Majorana edge mode, while there are four gapless modes which stem from four nodes of the $d_{x^2-y^2}$ -wave superconducting gap. To generate an energy gap in the Majorana spectrum, we need even numbers of Majorana modes which are paired into complex fermions. Thus, for this geometry, the chiral Majorana edge mode is stable against interactions with nodal excitations and also against impurity scattering. However, this argument is not applicable to the case with two open boundaries at the opposite sides of the system. In this case, two counterpropagating chiral Majorana modes reside in the two opposite edges, as depicted in Fig. 1(c). Interactions between bulk gapless nodal excitations and two chiral Majorana modes may give rise to long-range tunneling between two Majorana modes. We note that such long-range tunneling via nodal excitations does not occur in a clean system because of the mismatch of the Fermi momenta of nodal excitations $k_F \neq 0$ and that of the chiral

Majorana modes with which $k_F \sim 0$ for the $d_{x^2-y^2}$ -wave pairing. When there are impurity potentials, a Majorana mode and nodal excitations on an edge can be hybridized via impurity scattering, leading to the long-range tunneling of the MFs in two opposite edges: $\mathcal{H}_{\text{tun}} = t(\mathbf{r}_0 - \mathbf{r}'_0) i \gamma(0, y_0) \gamma(L, y'_0)$, where $\gamma(x, y)$ is a MF operator, $\mathbf{r}_0 = (0, y_0)$ and $\mathbf{r}'_0 = (L, y'_0)$ are the positions of impurities, and the tunneling amplitude $t(\mathbf{r}_0 - \mathbf{r}'_0) \sim 1/|\mathbf{r}_0 - \mathbf{r}'_0|$ for a large $|\mathbf{r}_0 - \mathbf{r}'_0|$. We introduce a complex fermion operator: $\alpha(y) = \frac{1}{2}[\gamma(0, y) + i\gamma(L, -y + y_0 + y'_0)]$. Then, the Hamiltonian for two chiral Majorana edge states can be rewritten into that of the 1D chiral Dirac fermion [30]: $\mathcal{H}_{\text{edge}} = -iv \int dy \alpha^\dagger(y) \partial_y \alpha(y)$. The long-range tunneling term is also expressed as $\mathcal{H}_{\text{tun}} = t_0 [2\alpha^\dagger(y_0)\alpha(y_0) - 1]$. Because of the chiral character of the Dirac fermion α , this tunneling term raises only forward scattering, the effect of which is merely to shift the chemical potential. As a result, the chiral Dirac fermion is still gapless. Going back to the Majorana fields, we conclude that the two chiral Majorana edge modes are stable against sufficiently dilute impurities. In contrast, in the case of the d_{xy} -wave pairing, the long-range tunneling via nodal excitations exists even in the absence of impurities, as depicted in Fig. 1(f). In this case, an energy gap opens around $k_y \sim 0$, and the Majorana mode disappears even for a relatively large value of L , for which the Majorana mode still exists for the $d_{x^2-y^2}$ -wave pairing [see Figs. 1(b) and 1(e)].

We now consider the stability of the Majorana mode in a vortex core. In addition to the localized zero energy Majorana solution, there are also delocalized states caused by gapless nodal excitations. When there are multiple vortices in the system under consideration, these delocalized states raise long-range tunneling between spatially separated vortices, which may destroy the zero energy Majorana mode. In the system with odd numbers of vortices, one Majorana mode in a vortex core survives. However, in the case with even numbers of vortices, the Majorana mode disappears unless they are separated enough from each other.

Topological order and topological protection of Majorana fermions.—The above consideration strongly implies that there is a topological order which ensures the stability of MF modes even for nodal superconductors with bulk gapless excitations. However, in sharp contrast to a gapful topological order, the bulk Chern number ν_{Ch} is not well-defined for our gapless system. Nevertheless, we clarify here that *the parity of the Chern number $(-1)^{\nu_{\text{Ch}}}$ is well-defined even for nodal superconductors*. The parity of the Chern number ensures the stability of the topological order in our system.

Let us first try to define the Chern number in our gapless system. The simplest way to do this is to introduce a small perturbation eliminating all nodes (i.e., gapless points) in the spectrum. For instance, adding a small id_{xy} term in the gap function, we can easily remove all the nodes in our $d_{x^2-y^2}$ superconductor. After removing the nodal points,

the Chern number can be evaluated in the standard manner. This procedure, however, does not work well after all. The problem is that the value of the Chern number depends on the perturbation we choose. As a result, one cannot have a unique definition of the Chern number for gapless systems.

On the other hand, we find that this procedure does define the parity of the Chern number uniquely. From the particle-hole symmetry, the parity of the Chern number is recast into

$$(-1)^{\nu_{\text{Ch}}} = \exp \left[i \int_{\Gamma_1}^{\Gamma_2} dk_i A_i(\mathbf{k}) + i \int_{\Gamma_3}^{\Gamma_4} dk_i A_i(\mathbf{k}) \right], \quad (2)$$

where $A_i(\mathbf{k})$ is the ‘‘gauge field’’ defined by the bulk band wave function $|u_n(\mathbf{k})\rangle$, $A_i(\mathbf{k}) = i \sum_n \langle u_n(\mathbf{k}) | \partial_{k_i} u_n(\mathbf{k}) \rangle$, and Γ_i is the time-reversal invariant \mathbf{k} points, $\Gamma_{i=1,2,3,4} = (0, 0)$, $(\pi, 0)$, $(0, \pi)$, (π, π) [31]. Then, for the Hamiltonian (1), we can show that

$$(-1)^{\nu_{\text{Ch}}} = \prod_{i=1,2,3,4} \text{sgn}[\epsilon_{\Gamma_i}^2 + \Delta_{\Gamma_i}^2 - h^2], \quad (3)$$

irrespective of the perturbation (such as id_{xy} term) we choose [32]. This means that we have a unique value of the parity in the limit of $id_{xy} \rightarrow 0$; i.e., the parity of the Chern number $(-1)^{\nu_{\text{Ch}}}$ is well-defined even for nodal superconductors, although the Chern number ν_{Ch} itself is not. The parity of the Chern number characterizes the topological phase in nodal superconductors. For $(-1)^{\nu_{\text{Ch}}} = -1$, there exists an odd number of topologically stable MFs in the edges and in a vortex core for nodal superconductors. For example, for the model (1) with $-4t - \mu < h < -\mu$, we obtain $(-1)^{\nu_{\text{Ch}}} = -1$ from (3). Thus, the existence of the gapless Majorana edge mode in Fig. 1 is characterized by the odd parity $(-1)^{\nu_{\text{Ch}}} = -1$. On the other hand, for $(-1)^{\nu_{\text{Ch}}} = 1$, there is no topologically stable MF. We emphasize that the formula (3) is applicable only to systems with particle-hole symmetry, and, thus, the topological order in gapless systems is specific to topological superconducting states.

In addition to the parity of the Chern number, one can consider another topological number dubbed the 1D Z_2 invariant [31]. The 1D Z_2 invariant $(-1)^{\nu[C_{ij}]}$ is introduced as a line integral along a specific time-reversal invariant path C_{ij} passing through Γ_i and Γ_j . In a similar manner above, it is shown that the 1D Z_2 invariant is well-defined even for our nodal superconductor, and we obtain $(-1)^{\nu[C_{ij}]} = \text{sgn}[\epsilon_{\Gamma_i}^2 + \Delta_{\Gamma_i}^2 - h^2] \text{sgn}[\epsilon_{\Gamma_j}^2 + \Delta_{\Gamma_j}^2 - h^2]$. For the model (1) with $-4t - \mu < h < -\mu$, this formula yields $(-1)^{\nu[C_{12}]} = -1$ for both $d_{x^2-y^2}$ and d_{xy} superconductors. From the bulk-edge correspondence, this Z_2 invariant determines the location of the Majorana edge fermions at $k_y \sim 0$, as illustrated in Figs. 1(a), 1(b), and 1(d). Since the 1D Z_2 invariant is associated with a local structure in the BZ, its nontriviality does not directly lead to the topological stability. However, the 1D Z_2 invariant is useful for identifying the location of

zero energy Majorana edge modes, as shown above. On the other hand, the odd parity of the Chern number introduced above definitely characterizes the global nontrivial topology of the Hilbert space, ensuring the topological protection mechanism of Majorana modes.

The above consideration can be straightforwardly generalized to a general multiband nodal superconductor. In this case, when the superconducting gap vanishes or becomes sufficiently small at the time-reversal invariant \mathbf{k} points, Γ_i , the parity of the Chern number, is evaluated as $(-1)^{\nu_{\text{Ch}}} = \prod_{\alpha,i=1,2,3,4} \text{sgn}[\mathcal{E}_{\alpha}(\Gamma_i)]$, where $\mathcal{E}_{\alpha}(\mathbf{k})$ is the normal dispersion of the superconductor. The index α specifies an energy band including the spin degrees of freedom. Therefore, when the Fermi level is located close to odd numbers of time-reversal invariant \mathbf{k} points, and the superconducting gap vanishes at these points because of the symmetry requirement, the NCS nodal superconductor possesses topologically protected MF modes under an applied small magnetic field.

Experimental detection of Majorana fermions.—For the experimental detection of MFs in nodal superconductors, one promising approach is to exploit an interferometry measurement proposed for a superconductor-topological-insulator junction in Refs. [34,35]. We consider a setup similar to those proposals but with a difference that, instead of a superconductor-topological-insulator junction, a bulk d -wave Rashba superconductor is used. The contribution from nodal excitations to the conductance in the d -wave pairing state vanishes like $\sim T$ at sufficiently low temperatures, and, thus, the current is dominated by that carried by two Majorana edge modes. The dependence of the conductance on the parity of vorticity inside the superconductor signifies clearly the MF contributions [34,35]. Possible candidates of our proposal are heavy fermion NCS superconductors CeRhSi₃ and CeIrSi₃, in which the Fermi level is close to time-reversal invariant \mathbf{k} points at K points in the BZ [20,21,24]. Although there are two K points, one of them can be moved away from the Fermi level by applying an uniaxial strain in the x (or y) direction. Then, the non-Abelian topological order is realized. The non-Abelian nodal superconductor is also realizable in an interface between a centrosymmetric nodal superconductor such as high- T_c cuprates and a semiconductor, as considered in the case of the s -wave pairing state by Sau *et al.* and Alicea [15,16]. In such a system, because of the considerably large superconducting gap, the experimental detection of Majorana modes may be easier.

Summary.—We have demonstrated that even in nodal superconductors, MF modes, which are topologically protected against weak perturbations and lead to the non-Abelian statistics, are realized under a certain realistic condition, in spite of the existence of bulk gapless nodal excitations. Our results establish a concept of a gapless

topological phase and open the possibility of detecting MFs in various NCS superconductors with gap nodes.

This work is supported by KAKENHI from MEXT of Japan [Grants No. 19052003, No. 21102510, No. 22540383, and No. 22103005 (Innovative Areas “Topological Quantum Phenomena”)].

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