Exact and Explicit Probability Densities for One-Sided Lévy Stable Distributions

K. A. Penson[1,*] and K. Górska[1,2,†]

[1]Laboratoire de Physique Théorique de la Matière Condensée (LPTMC), Université Pierre et Marie Curie, CNRS UMR 7600, Tour 13 - 5ième étage, Boîte Courrier 121, 4 place Jussieu, F 75252 Paris Cedex 05, France

[2]Nicolaus Copernicus University, Institute of Physics, ulica Grudziądzka 5/7, PL 87-100 Toruń, Poland

(Received 7 July 2010; published 17 November 2010)

We study functions $g_\alpha(x)$ which are one-sided, heavy-tailed Lévy stable probability distributions of index α, $0 < \alpha < 1$, of fundamental importance in random systems, for anomalous diffusion and fractional kinetics. We furnish exact and explicit expressions for $g_\alpha(x)$, $0 \le x < \infty$, for all $\alpha = l/k < 1$, with k and l positive integers. We reproduce all the known results given by $k \le 4$ and present many new exact solutions for $k > 4$, all expressed in terms of known functions. This will allow a "fine-tuning" of α in order to adapt $g_\alpha(x)$ to a given experimental situation.

DOI: 10.1103/PhysRevLett.105.210604 PACS numbers: 05.40.Fb, 02.50.Ng

Theoretical description of many collective physical systems which includes a special sort of disorder or randomness often requires a radical departure from classical diffusive behavior. On the probabilistic level this signifies the appearance of distributions with nonconventional characteristics, such as diverging mean and variance along with all integer moments different from the zeroth one. In this context the discovery of particular distributions with these properties plays the dominant role. They are now called the Lévy stable laws [1], whose generic example is $g_{1/2}(x) = (2\sqrt{\pi}x^{3/2})^{-1}\exp(-1/4x)$, $x \ge 0$; the word "stable" means here that the product of characteristic functions (CF) of two such laws is a CF of another law of the same type [1]. The general distribution of that type $g_\alpha(x)$ can be shown to possess the CF or the Laplace transform of the form [2–4]

$$\int_0^\infty e^{-px} g_\alpha(x)dx = e^{-p^\alpha}, \qquad p > 0, 0 < \alpha < 1, \quad (1)$$

which is the well-known Kohlrausch-Williams-Watts function [5] or stretched exponential. Several independent proofs can be given that $g_\alpha(x)$ obeying Eq. (1) is positive [2,3,6].

The functions $g_\alpha(x)$ are ubiquitous in many fields of condensed and soft matter physics [7–9], geophysics [10], meteorology [11], economics [12], fractional kinetics [13,14], etc. For instance, the value $\alpha = 1/4$ is thought to describe mechanical and dielectric properties of glassy polymers [15]. It is also confirmed that the same value of α is relevant for a statistical description of subrecoil laser cooling [16,17]. In general, numerous phenomena falling in the class of subdiffusion [18] call for $g_\alpha(x)$, $\alpha < 1$, in their theoretical description. On the theoretical side, the Lévy stable distributions are essential tools in the study of random maps and resulting combinatorial structures [19]. The actual use of Lévy stable type distributions has been hampered for subjective and objective reasons [20,21]. The subjective ones include a certain reticence to use distributions with both mean and variance diverging. The main objective reason is a lack of knowledge of $g_\alpha(x)$ for most values of α. The existing interpolation formulas [15] appear to be cumbersome to use.

It seems that obtaining explicit $g_\alpha(x)$ for arbitrary $0 < \alpha < 1$ constitutes a true challenge: the explicit forms of $g_\alpha(x)$ are known only for a limited number of values of α, i.e., $\alpha = 1/2$ [see $g_{1/2}(x)$ above], 1/4 [22], 1/3 [23], 2/3 [24], and 3/4 [23]. The formal solution for arbitrary α [4,9] is only of limited use as it requires series or asymptotic expansions, which may become problematic, especially for small α.

The objective of this work is to present a universal formula for $g_\alpha(x)$, $\alpha = l/k$, with $k > l$ positive integers, which is exact and explicit. It reproduces all the known cases enumerated above, and yields an infinity of new solutions for $k > 4$, of which we quote, for the first time, several instances.

Equation (1) for $\alpha = l/k$ can be inverted giving

$$g_{l/k}(x) = \frac{\sqrt{kl}}{(2\pi)^{(k-l)/2}} \frac{1}{x} G^{k,0}_{l,k}\left(\frac{l^l}{k^k x^l} \,\middle|\, \begin{matrix}\Delta(l,0)\\ \Delta(k,0)\end{matrix}\right), \quad (2)$$

valid for all $x \ge 0$, where $G^{m,n}_{p,q}(z|^{(a_p)}_{(b_q)})$ is the Meijer G function [25,26] and $\Delta(k,a) = \frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k-1}{k}$ is a special list of k elements. Equation (2) is listed without proof as a special case for $\nu = 0$ and $a = 1$ of formula 2.2.1.19, in Vol. 5 of [26]. The detailed demonstration of Eq. (2) with a combined use of Laplace and Mellin transforms will be given elsewhere. It turns out that the right-hand side of Eq. (2) is a finite sum of $k-1$ generalized hypergeometric functions of type ${}_pF_q(^{(a_p)}_{(b_q)}|z)$ [26]:

$$g_{l/k}(x) = \sum_{j=1}^{k-1} \frac{b_j(k,l)}{x^{1+j\frac{l}{k}}} {}_{l+1}F_k\left(\begin{matrix}1, \Delta(l, 1+jl/k)\\ \Delta(k, j+1)\end{matrix} \,\middle|\, (-1)^{k-l}\frac{l^l}{k^k x^l}\right), \quad (3)$$

where $b_j(k,l)$ are numerical coefficients given by

$$b_j(k,l) = \frac{l^{jl/k}\sqrt{kl}}{k^j(2\pi)^{(k-l)/2}} \frac{\left[\prod_{i=1}^{j-1}\Gamma\left(\frac{i-j}{k}\right)\right]\left[\prod_{i=j+1}^{k-1}\Gamma\left(\frac{i-j}{k}\right)\right]}{\prod_{i=1}^{l-1}\Gamma\left(\frac{i}{l}-\frac{j}{k}\right)}, \tag{4}$$

where $\Gamma(y)$ is Euler's gamma function. Equation (3) is the exact implementation of the program outlined in the fundamental work of Scher and Montroll [23], in which it was conjectured that $g_{l/k}(x)$ can be expressed in terms of ${}_pF_q$'s. However, in [23] actually only one new instance $g_{3/4}(x)$ was written down. Our formula Eq. (3), after appropriate reductions in ${}_pF_q$'s, see below, gives all exactly known cases mentioned above [20–23], with $g_{1/4}(x)$ [22],

$$\begin{aligned} g_{1/4}(x) = {} & \frac{b_1(4,1)}{x^{5/4}}\,{}_0F_2\left(\begin{matrix} - \\ 1/2,\,3/4 \end{matrix}\,\middle|\,\frac{-1}{4^4 x}\right) \\ & + \frac{b_2(4,1)}{x^{3/2}}\,{}_0F_2\left(\begin{matrix} - \\ 3/4,\,5/4 \end{matrix}\,\middle|\,\frac{-1}{4^4 x}\right) \\ & + \frac{b_3(4,1)}{x^{7/4}}\,{}_0F_2\left(\begin{matrix} - \\ 5/4,\,3/2 \end{matrix}\,\middle|\,\frac{-1}{4^4 x}\right), \end{aligned} \tag{5}$$

and offers an unlimited number of new solutions for $g_{l/k}(x)$, $k > 4$, e.g.,

$$g_{p/5}(x) = \sum_{j=1}^{4} \frac{b_j(5,p)}{x^{1+jp/5}}\,{}_{p+1}F_5\left(\begin{matrix} 1,\,\Delta(p, 1+jp/5) \\ \Delta(5, j+1) \end{matrix}\,\middle|\,\frac{p^p}{5^5 x^p}\right), \tag{6}$$

$p = 1, \ldots, 4$; see Table I for coefficients in Eqs. (5) and (6), etc.

The symbol $\Delta(k,a)$ in Eq. (3) permits one to encode all the possible cases of k and l in a single formula. However, we draw attention to the fact that cancellations will appear there due to the obvious identity

$${}_{p+r}F_{q+r}\left(\begin{matrix} (a_p),\,(\alpha_r) \\ (b_q),\,(\alpha_r) \end{matrix}\,\middle|\,x\right) = {}_pF_q\left(\begin{matrix} (a_p) \\ (b_q) \end{matrix}\,\middle|\,x\right),$$

where (α_r) is an arbitrary sequence of r parameters not equal to zero or to negative integers. Thus Eq. (5) is a sum of three ${}_0F_2$ functions, and likewise $g_{3/4}(x)$, which is not specified here, will be a sum of three ${}_2F_2$ functions, neatly confirming Eq. (C10) of [23], etc. In this manner for any l/k, a closed form of $g_{l/k}(x)$ can be obtained from Eq. (3). However, only for $k \leq 3$ can it be written down in terms of standard special functions [5,20,21,23,24].

A heuristic indication of how Eq. (3) comes about can be obtained from the series representation for $g_\alpha(x)$ derived by Humbert [27], discussed, for example, by Hughes [28] and used in [19]. The series

$$g_\alpha(x) = \frac{1}{\pi}\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j!\,x^{1+\alpha j}}\Gamma(1+\alpha j)\sin(\pi\alpha j) \tag{7}$$

is a convergent expansion valid for all $0 < \alpha < 1$ and $x > 0$. For $\alpha = l/k$ the decomposition of the summation index j modulo k yields an equivalent representation of $g_{l/k}(x)$ as a sum of $k-1$ infinite series. The structure of the coefficients in each of these series involves gamma function ratios of type $\Gamma(li+\theta)/\Gamma(ki+\phi)$, where i is the new summation index and θ and ϕ are simple functions of k and l. The application of the Gauss-Legendre multiplication formula to both of these gamma functions allows the identification of ${}_pF_q$'s in Eq. (3) and the extraction of coefficients $b_j(k,l)$ of Eq. (4).

The advantage of our solution, Eqs. (3) and (4) over Eq. (7) is clearly seen in practice, in conjunction with the use of computer algebra systems [29]. Since in recent versions of these systems the hypergeometric functions ${}_pF_q$ as well as the Meijer G function are fully implemented, their use permits high-precision calculations. For the reader's convenience we give in [30] the MAPLE® syntax for $g_{l/k}(x)$; see Eq. (2) above. Our experience indicates that for small α our results for small x are more practical to use than the $x \to 0$ asymptotics given in [3]. The reason is that there the region of applicability of Mikusiński's asymptotic expansion [3] shrinks to exceedingly small values of x. For example, for $\alpha = 1/20$, $g_{1/20}(0) = 0$, but a huge peak in $g_{1/20}(x)$ appears already at $x \sim 10^{-14}$. In contrast, our formulas work fine for any x in this region. In the opposite limit for $\alpha \lesssim 1$ the Humbert expansion Eq. (7) is slowly convergent for $x < \alpha$, but

TABLE I. Coefficients of Eqs. (5) and (6); $A = \sin(\pi/5)$ and $B = \sin(2\pi/5)$.

j	1	2	3	4
$b_j(4,1)$	$\frac{1}{4\Gamma(\frac{3}{4})}$	$\frac{-1}{4\sqrt{\pi}}$	$\frac{\sqrt{2}\Gamma(\frac{3}{4})}{16\pi}$	$\cdots$
$b_j(5,1)$	$\frac{\sqrt{5}\Gamma(\frac{1}{5})}{20\pi B}$	$\frac{-\sqrt{5}\Gamma(\frac{2}{5})}{20\pi A}$	$\frac{\sqrt{5}\Gamma(\frac{3}{5})}{40\pi A}$	$\frac{-\sqrt{5}\Gamma(\frac{4}{5})}{120\pi B}$
$b_j(5,2)$	$\frac{\sqrt{5}\times 2^{2/5}\Gamma(\frac{1}{5})}{10\sqrt{\pi}\Gamma(\frac{3}{10})B}$	$\frac{-\sqrt{5}\times 2^{4/5}\Gamma(\frac{2}{5})}{10\sqrt{\pi}\Gamma(\frac{1}{10})A}$	$\frac{-\sqrt{5}\times 2^{1/5}\Gamma(\frac{3}{5})}{100\sqrt{\pi}\Gamma(\frac{9}{10})A}$	$\frac{\sqrt{5}\times 2^{3/5}\Gamma(\frac{4}{5})}{100\sqrt{\pi}\Gamma(\frac{7}{10})B}$
$b_j(5,3)$	$\frac{3\sqrt{5}\times 3^{1/10}\Gamma(\frac{1}{5})}{10\Gamma(\frac{2}{15})\Gamma(\frac{7}{15})B}$	$\frac{\sqrt{5}\times 3^{7/10}\Gamma(\frac{2}{5})}{50\Gamma(\frac{4}{15})\Gamma(\frac{14}{15})A}$	$\frac{-3\sqrt{5}\times 3^{3/10}\Gamma(\frac{3}{5})}{25\Gamma(\frac{1}{15})\Gamma(\frac{11}{15})A}$	$\frac{-7\sqrt{5}\times 3^{9/10}\Gamma(\frac{4}{5})}{750\Gamma(\frac{8}{15})\Gamma(\frac{13}{15})B}$
$b_j(5,4)$	$\frac{4\times 2^{1/10}5^{-1/2}\sqrt{\pi}\Gamma(\frac{1}{5})}{\Gamma(\frac{3}{10})\Gamma(\frac{1}{20})\Gamma(\frac{11}{20})B}$	$\frac{6\times 2^{7/10}5^{-3/2}\sqrt{\pi}\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{10})\Gamma(\frac{7}{20})\Gamma(\frac{17}{20})A}$	$\frac{14\times 2^{3/10}5^{-5/2}\sqrt{\pi}\Gamma(\frac{3}{5})}{\Gamma(\frac{9}{10})\Gamma(\frac{3}{20})\Gamma(\frac{13}{20})A}$	$\frac{11\times 2^{9/10}5^{-7/2}\sqrt{\pi}\Gamma(\frac{4}{5})}{\Gamma(\frac{7}{10})\Gamma(\frac{9}{20})\Gamma(\frac{19}{20})B}$

approximation [3] works well as then $g_\alpha(x)$ is very close to zero in a considerable region near $x = 0$ [e.g., already for $\alpha = 5/6$ the function $g_{5/6}(x)$ is practically equal to zero up to $x \approx 0.35$]. Such a practically flat region for small x can also be seen for $\alpha = 4/5$; compare curve III on Fig. 3.

In Fig. 1 we compare three distributions for $l/k = 1/2$, $1/3$, and $1/4$. The salient feature for $l/k = 1/4$ is the appearance of a sharp maximum for very small x so that these three curves can be barely shown on the same scale. Analogously, for $l/k = 1/5$ the maximum of $g_{1/5}(x)$ appears at $x_0(1/5) \approx 0.0002$ and the value $g_{1/5}(x_0) \approx 25$. For $x < x_0(1/5)$ the values of $g_{1/5}(x)$ are very close to zero. As already mentioned above, for smaller values of l/k this type of behavior is even more pronounced and it explains *a posteriori* the difficulties encountered in devising approximations valid for small l/k and small x [15,24]. In Fig. 2 we present the comparison of several distributions for values $l/k \approx 1/2$. Here the "sharpening" of the distributions, as l/k goes from $1/2$ to smaller values, is very clearly visible but is less dramatic than in Fig. 1. We present in Fig. 3 the new distributions $g_{p/5}(x)$ given by Eq. (6) for $p = 2$, 3, and 4.

All these probability distributions share the following features: (a) $g_\alpha(x) \to 0$, for $x \to 0$, where they present an essential singularity $\sim x^{[(-2+\alpha)/2(1-\alpha)]} \exp[-A(\alpha)x^{-\alpha/(1-\alpha)}]$, $A(\alpha) > 0$ [3]; (b) $g_\alpha(x) \to B(\alpha)x^{-(1+\alpha)}$, for $x \to \infty$, $B(\alpha) > 0$, indicating heavy-tailed asymptotics for large x; (c) all their fractional moments $M_\alpha(\mu) = \int_0^\infty x^\mu g_\alpha(x)dx = \Gamma(-\mu/\alpha)/[\alpha\Gamma(-\mu)]$, for real μ, $-\infty < \mu < \alpha$, including $M_\alpha(0) = 1$, are finite, and are infinite otherwise; (d) $g_\alpha(x)$ are unimodal with the maximum at $x_0(\alpha)$, and $x_0(\alpha) \to 0$ as $\alpha \to 0$.

The distributions $g_\alpha(x)$ constitute basic ingredients of all theories of anomalous diffusion where they are employed to produce solutions $P_\alpha(x, t)$ in the space-time domain of various forms of the Fokker-Planck equations along with their fractional generalizations [6,22]. For instance, in [22] $P_\alpha(x, t)$ is given as a convolution (called there inverse Lévy transform) of $\frac{d}{ds}[-g_\alpha(t/s^{1/\alpha})]$ with $P_1(x, t)$ being a normalized solution of the ordinary Fokker-Planck equation; see Eq. (1) in [22]. The explicit forms of $g_\alpha(x)$ presented here will permit further development of this ambitious approach.

The availability of $g_{l/k}(x)$ makes it possible to fully describe the long tail distributions of carrier transit times in amorphous materials such as As_2Se_3 and trinitrofluorenone and polyvinylcarbazole (TNF-PVK). In fact, in classic work [23] the measured values of α for these two materials were $\alpha = 0.45$ and $\alpha = 0.8$ respectively; compare Fig. 6 of [23]. These values were for a long time intractable theoretically. From now on, setting $\alpha = 9/20$ and $\alpha = 4/5$ in our Eqs. (2) and (3) directly provides the sought-for framework for interpretation of these data. The appropriate distributions are presented as curve II in Fig. 2 ($\alpha = 9/20$) and curve III in Fig. 3 ($\alpha = 4/5$).

We believe that the exact forms of $g_\alpha(x)$ obtained in this work, along with their asymptotics for $x \to \infty$ and exact values of fractional moments, constitute a solid basis to extract a value of α best suited for an experimental situation at hand. Once it has been done, such description can be further fine-tuned by choosing values of k and l which would optimize the choice of α. We hope that this approach will prove useful in practical applications.

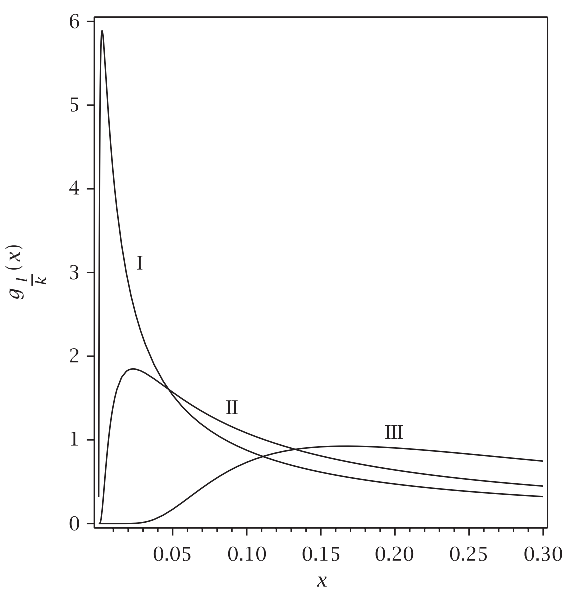

FIG. 1. Comparison of $g_{l/k}(x)$. Curves I, II, and III correspond to $l/k = 1/4$ [see Eq. (5)], 1/3, and 1/2, respectively.

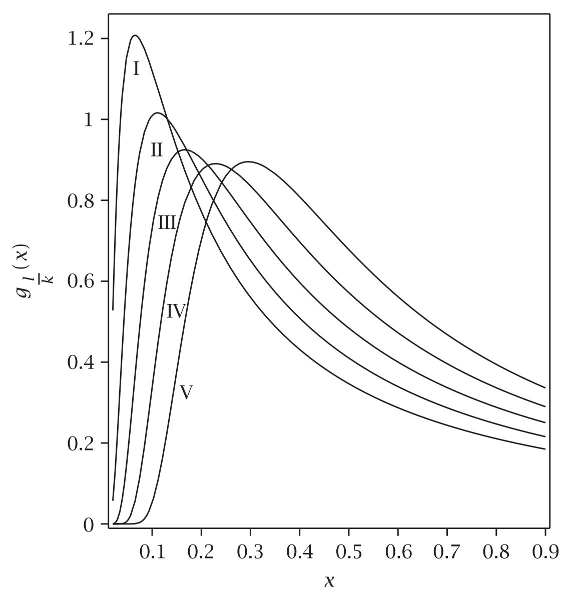

FIG. 2. Comparison of $g_{l/k}(x)$. Curves I, II, III, IV, and V correspond to $l/k = 2/5$, 9/20, 1/2, 11/20, and 3/5, respectively. Calculations were performed using Eqs. (3) and (4).

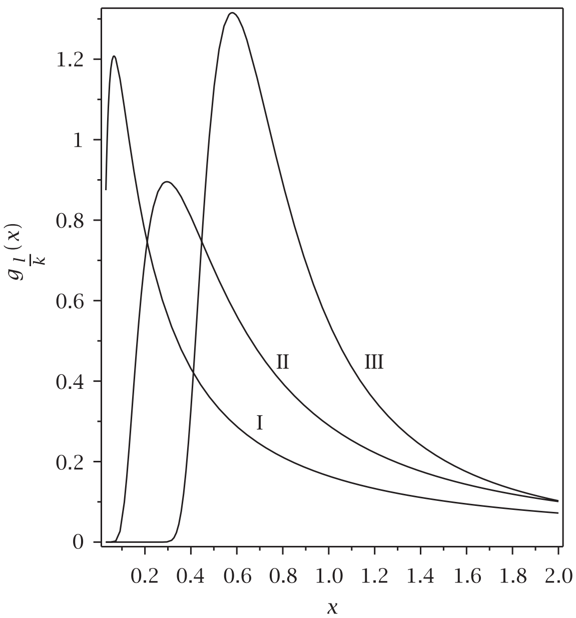

FIG. 3. Comparison of $g_{l/k}(x)$. Curves I, II, and III correspond to $l/k = 2/5$, $3/5$, and $4/5$, respectively. Calculations were performed using Eq. (6).

The authors acknowledge support from Agence Nationale de la Recherche (Paris, France) under Program No. ANR-08-BLAN-0243-2. We thank Professor E. Barkai for kindly informing us about his results obtained in [22].

*penson@lptl.jussieu.fr
†kasia_gorska@o2.pl

[1] J.-P. Kahane, in *Lévy Flights and Related Topics in Physics*, edited by M.F. Shlesinger, G.M. Zaslavsky, and U. Frisch, Lecture Notes in Physics Vol. 450 (Springer, Berlin, 1995).
[2] H. Pollard, Bull. Am. Math. Soc. **52**, 908 (1946).
[3] J. Mikusiński, Stud. Math. **18**, 191 (1959).
[4] W.R. Schneider, in *Stochastic Processes in Classical and Quantum Systems*, edited by S. Albeverio, G. Casati, and D. Merlini, Lecture Notes in Physics Vol. 262 (Springer, Berlin, 1986).
[5] R.S. Anderssen, S.A. Husain, and R.J. Loy, ANZIAM J. **45**, C800 (2004).
[6] I.M. Sokolov, Phys. Rev. E **63**, 011104 (2000).
[7] R. Metzler and J. Klafter, J. Phys. A **37**, R161 (2004).
[8] P.-G. de Gennes, Macromolecules **35**, 3785 (2002).
[9] R. Hilfer, Phys. Rev. E **65**, 061510 (2002).
[10] O. Sottolongo-Costa, J.C. Antoranz, A. Posadas, F. Vidal, and A. Vazquez, Geophys. Res. Lett. **27**, 1965 (2000).
[11] M. Lagha and M. Bensebti, arXiv:00194153.
[12] R.N. Mantegna and H.E. Stanley, Nature (London) **376**, 46 (1995).
[13] I.M. Sokolov, J. Klafter, and A. Blumen, Phys. Today, No. 11, **55**, 48 (2002).
[14] A.V. Chechkin, V. Yu. Gonchar, R. Gorenflo, N. Korabel, and I.M. Sokolov, Phys. Rev. E **78**, 021111 (2008).
[15] J.T. Bendler, J. Stat. Phys. **36**, 625 (1984).
[16] F. Bardou, J.-P. Bouchaud, O. Emile, A. Aspect, and C. Cohen-Tannoudji, Phys. Rev. Lett. **72**, 203 (1994).
[17] F. Bardou, J.-P. Bouchaud, A. Aspect, and C. Cohen-Tannoudji, *Lévy Statistics and Laser Cooling* (Cambridge University Press, Cambridge, England, 2002).
[18] T. Koren, J. Klafter, and M. Magdziarz, Phys. Rev. E **76**, 031129 (2007).
[19] P. Flajolet and R. Sedgewick, *Analytic Combinatorics* (Cambridge University Press, Cambridge, England, 2009).
[20] V.V. Uchaĭkin, Zh. Eksp. Teor. Fiz. **115**, 2113 (1999) [Sov. Phys. JETP **88**, 1155 (1999)].
[21] V.V. Uchaĭkin, Usp. Fiz. Nauk **173**, 847 (2003) [Sov. Phys. Usp. **46**, 821 (2003)].
[22] E. Barkai, Phys. Rev. E **63**, 046118 (2001).
[23] H. Scher and E.W. Montroll, Phys. Rev. B **12**, 2455 (1975).
[24] E.W. Montroll and J.T. Bendler, J. Stat. Phys. **34**, 129 (1984).
[25] O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions. Theory and Algorithmic Tables* (Ellis Horwood, Chichester, England, 1983).
[26] A.P. Prudnikov, Yu., A. Brychkov, and O.I. Marichev, *Integrals and Series* (Gordon and Breach, Amsterdam, 1998), Vols. 1–5.
[27] P. Humbert, Bull. Soc. Math. Fr. **69**, 121 (1945).
[28] B.D. Hughes, *Random Walks and Random Environments* (Clarendon Press, Oxford, 1995), Vol. 1.
[29] We have made extensive use of MAPLE® in this work.
[30] Here is the MAPLE® procedure LevyDist (k, l, x) used to calculate Eq. (2): LevyDist := proc(k, l, x) simplify(convert(sqrt$(k*l)*$MeijerG([[], [seq$(j1/l, j1 = 0..l-1)$]], [[seq$(j2/k, j2 = 0..k-1)$], []], l^$l/(k$^$k*x$^$l)$)/$(x*(2*Pi)$^$((k-l)/2)$), StandardFunctions)); end; Analogous syntax can be given for MATHEMATICA®.