Anderson Localization in Quark-Gluon Plasma

Tamás G. Kovács and Ferenc Pittler
Department of Physics, University of Pécs, H-7624 Pécs Ifjúság útja 6, Hungary
(Received 29 June 2010; published 2 November 2010)

At low temperature the low end of the QCD Dirac spectrum is well described by chiral random matrix theory. In contrast, at high temperature there is no similar statistical description of the spectrum. We show that at high temperature the lowest part of the spectrum consists of a band of statistically uncorrelated eigenvalues obeying essentially Poisson statistics and the corresponding eigenvectors are extremely localized. Going up in the spectrum the spectral density rapidly increases and the eigenvectors become more and more delocalized. At the same time the spectral statistics gradually crosses over to the bulk statistics expected from the corresponding random matrix ensemble. This phenomenon is reminiscent of Anderson localization in disordered conductors. Our findings are based on staggered Dirac spectra in quenched lattice simulations with the $SU(2)$ gauge group.

DOI: 10.1103/PhysRevLett.105.192001 PACS numbers: 12.38.Mh, 11.15.Ha, 11.30.Rd, 12.38.Gc

The spectrum of the QCD Dirac operator contains important information regarding the properties of strongly interacting physical systems. The statistical properties of the spectrum completely determine the bulk thermodynamical observables. The low end of the Dirac spectrum is particularly important since that part dominates the quark propagators. The statistics of low Dirac eigenvalues is fundamentally different in the low temperature chirally broken phase and the high temperature chirally symmetric phase. According to the Banks-Casher formula [1] the spectral density around zero is proportional to the order parameter of chiral symmetry breaking. Therefore in the chirally symmetric phase it vanishes whereas in the broken phase it is nonzero. In the latter case random matrix theory (RMT) provides an essentially complete statistical description of the low lying part of the Dirac spectrum [2]. In the intermediate volume, so-called epsilon regime the RMT behavior is well understood analytically through an effective sigma model description and numerically through lattice simulations.

In contrast, above the finite temperature transition T_c in the chirally symmetric phase there is no well understood statistical description of the low eigenvalues of the Dirac operator. In this regime, lacking any analytical insight, one can regard the Dirac operator as a randomly fluctuating matrix of size going to infinity in the thermodynamic limit. From this perspective there are two fundamentally different possibilities for the spectrum of the Dirac operator. If typical random fluctuations can freely mix eigenvectors, eigenvectors become extended and the eigenvalue statistics is described by the corresponding RMT. If on the other hand, fluctuations in the matrix elements cannot mix the eigenvectors, in some basis they become localized and the eigenvalues become independent, obeying essentially Poisson statistics. Lattice simulations can test which scenario happens in reality.

Above T_c the spectral density vanishes at zero and RMT has predictions for the eigenvalue statistics at such a "soft edge" [3]. Lattice simulations, however, did not find agreement with these predictions [4,5]. Another earlier study [6] did not focus on the spectrum edge, but considered full Dirac spectra and found bulk RMT statistics for the full spectrum. More recently the possibility of Poisson eigenvalue statistics was suggested again in Ref. [7]. Based on lattice simulations the authors argued that around T_c the low temperature RMT statistics is gradually deformed towards Poisson statistics. Reference [8], based again on lattice simulations, found that although eigenmodes are localized above T_c, localization is most likely a finite volume artifact. This finding would disfavor the appearance of Poisson statistics in the spectrum. Recently, using overlap fermion lattice simulations, Poisson behavior was found for the lowest two eigenvalues [9].

In the present Letter we offer a better understanding of this rather unclear situation. We show that above T_c the lowest part of the spectrum consists of statistically independent eigenvalues obeying Poisson statistics and the corresponding eigenvectors are extremely localized. Going up in the spectrum the spectral density rapidly increases and the eigenvectors get delocalized. At the same time the spectral statistics gradually crosses over to the bulk statistics expected in the corresponding random matrix ensemble. We also show that the number of Poisson type eigenvalues depends only on the physical temperature and the physical spatial volume and not on the lattice spacing. The phenomenon we report here is analogous to Anderson localization occurring in crystalline conductors in the presence of disorder. In that case disorder causes the appearance of localized electron states at the band edge, but for sufficiently weak disorder states towards the band center remain delocalized [10]. The corresponding eigenvalue statistics changes from Poisson around the band edge to random matrix statistics towards the band center [11].

At first we summarize the details of the numerical simulations. The data are based on lattice simulations of

the $SU(2)$ gauge theory with Wilson plaquette coupling $\beta = 2.6$ and time extension $N_t = 4$. This corresponds to a temperature of $T = 2.6T_c$, well above the finite temperature phase transition. We used the quenched approximation (ignoring dynamical quarks) to make the scale setting through the critical temperature simple and unambiguous. The simulations were done at four different spatial volumes, $N_s^3 = 16^3, 24^3, 32^3, 48^3$. To assess what happens in the continuum limit we included an additional $N_t = 6$, $N_s = 36$, $\beta = 2.725$ ensemble matched to the $N_s = 24$ simulation in terms of physical box size and temperature, but on a 1.5 times finer lattice. On these ensembles we computed the 256 smallest positive eigenvalues of the staggered Dirac operator. Because of the exact twofold degeneracy of the eigenvalues this yields 128 independent positive eigenvalues per configuration that we used for the statistical analysis.

The low lying QCD Dirac spectrum is known to depend strongly on the temporal fermionic boundary condition through the lowest Matsubara frequency. The boundary condition is effectively a combination of the gauge parallel transporter along the closed time direction (Polyakov loop) and the explicitly chosen antiperiodic boundary condition [12]. In the quenched $SU(2)$ gauge theory the Polyakov loop $Z(2)$ symmetry is spontaneously broken above T_c. Although in the quenched theory the two sectors are equivalent, here we only use configurations in the ''physical'' Polyakov loop sector, the one that would survive in the presence of dynamical fermions. We use fermion boundary conditions that are antiperiodic in the time direction and periodic in all the spatial directions.

As a first step we studied the spatial localization of the low eigenmodes. A possible way to measure that is through the quantity

$$\mathcal{V}_\psi = \left[\sum_x (\psi^\dagger \psi(x))^2\right]^{-1}, \tag{1}$$

where ψ is a normalized eigenvector. $\mathcal{V}_\psi$ essentially measures the total volume occupied by ψ. This is seen by noting that in the special case when ψ is constant in a subvolume v of the total volume V and is zero elsewhere, $\mathcal{V}_\psi = v$ [13]. Assuming that at high temperature eigenmodes can maximally spread in the short time direction one can define a length scale (in lattice units)

$$d_\psi = \left[\frac{\mathcal{V}_\psi}{N_t}\right]^{1/3} \tag{2}$$

characterizing the spatial extension of the eigenvectors. In Fig. 1 we plot how this quantity changes in the spectrum. We computed the average eigenmode size d_ψ as a function of the corresponding eigenvalue for different spatial volumes. It is apparent that the lowest eigenmodes corresponding to eigenvalues $\lambda a < 0.22$ are very localized and their spatial extension is independent of the box size. Above that point the eigenvectors rapidly start to delocalize and their spatial size becomes dependent on the box size. This point in the spectrum is known as the mobility edge in the context of Anderson localization.

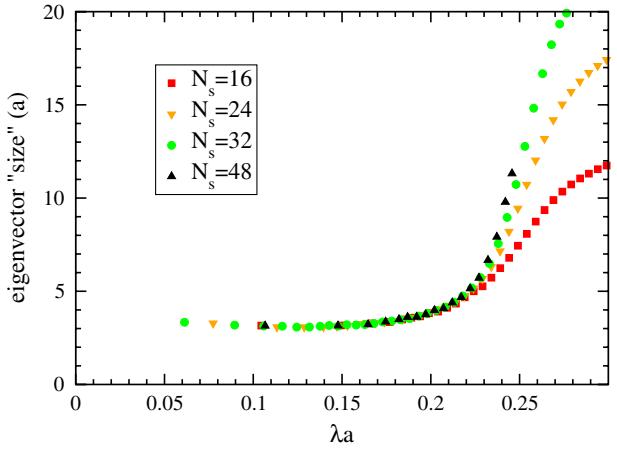

FIG. 1 (color online). The average linear extension of eigenvectors d_ψ as a function of the corresponding eigenvalues. The different symbols correspond to spatial box sizes $N_s = 16, 24, 32, 48$.

The extremely stable localization of the smallest modes with respect to increasing volume might seem to contradict Ref. [8], arguing that localization is a finite volume artifact. Note, however, that they define localization in terms of the inverse participation ratio and call a state localized essentially if it occupies a volume smaller than a fixed finite fraction of the total volume. In contrast, the modes we call localized occupy a fixed finite volume and therefore zero fraction of the total volume in the thermodynamic limit. Our definition of ''localized mode'' is thus more restrictive than that of [8] but conforms with the literature on Anderson localization where localization is understood to be exponential.

Besides the localization of eigenvectors the other important factor determining how easily eigenvectors can mix is the spectral density. A useful quantity that reflects the combined effect of localization and spectral density can be defined as follows. For each eigenvector ψ the participation ratio $\mathcal{V}_\psi/V$ is an approximate measure of the fraction of the total four-volume occupied by the eigenmode. We call the cumulative volume fill fraction the sum of the participation ratios of all the eigenvectors corresponding to eigenvalues less than λ. In Fig. 2 we show the volume fill fraction as a function of λ. In the lowest part of the spectrum where the spectral density is small and eigenvectors are localized the fill fraction is much smaller than unity. These eigenvectors can in principle be produced independently in different subvolumes, with no spatial overlap. If this is the case the corresponding eigenvalues are expected to be independently distributed and obey Poisson statistics. In contrast, above $\lambda a > 0.28$ the volume fill fraction is much bigger than unity and the eigenvectors here must strongly overlap. Therefore they can freely mix

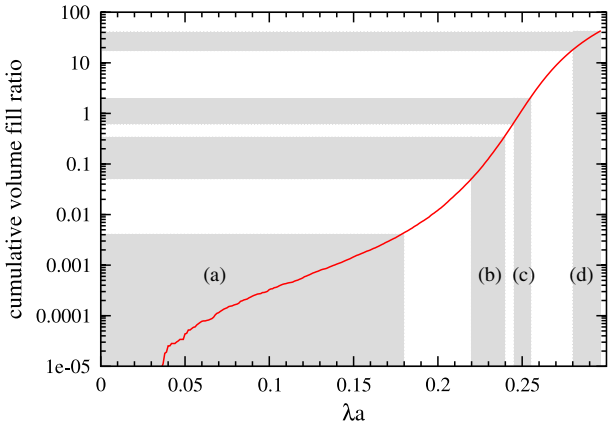

FIG. 2 (color online). The cumulative volume fill fraction for the $24^3 \times 4$ ensemble. The unfolded level spacing distribution will be computed separately in the four shaded spectral regions marked by (a)–(d) (see Fig. 3).

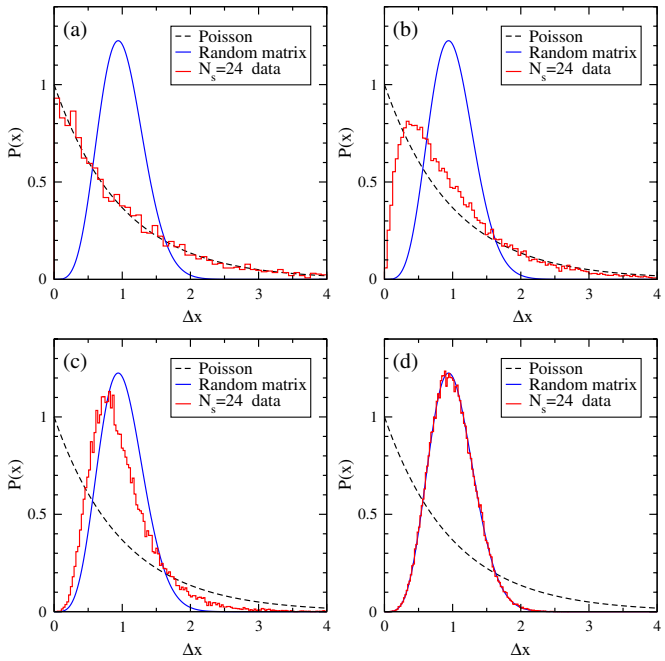

FIG. 3 (color online). The panels show the unfolded level spacing distribution in different regions of the spectrum. The labeling (a)–(d) corresponds to the regions indicated in Fig. 2 with the shaded areas. The curved lines are the exponential distribution and the Wigner surmise.

and from this point up in the spectrum the eigenvalue statistics is expected to be described by random matrix theory. Between the two extremes there can be a transition from Poisson to random matrix statistics.

A simple way of testing these expectations is to consider the unfolded level spacing distribution computed separately for eigenmodes in the above described regimes. Unfolding is a simple transformation of the eigenvalues commonly used in random matrix theory to extract universal spectral correlations that are independent of the spectral density. Numerically we unfolded by ordering all the eigenvalues by magnitude on all configurations in the given ensemble and mapping each eigenvalue to its rank order normalized by the number of configurations in the given ensemble. By construction the eigenvalues transformed in this way have constant unit spectral density.

If the original eigenvalues are uncorrelated, the unfolded level spacing distribution is expected to be a simple exponential $P(x) = e^{-x}$. In contrast, if the eigenvalues obey random matrix statistics the unfolded level spacing distribution should follow the so-called Wigner surmise of the corresponding random matrix ensemble [2]. The staggered Dirac operator with fermions in the fundamental representation of the $SU(2)$ gauge group correspond to the chiral symplectic random matrix ensemble and the Wigner surmise in that case is [14]

$$P(x) = \frac{2^{18}}{3^6\pi^3} x^4 \exp\left(-\frac{64}{9\pi} x^2\right). \tag{3}$$

In Fig. 3 we plot the unfolded level spacing distribution averaged separately for the four regions indicated in Fig. 2. For comparison we also show the exponential and Wigner surmise distributions expected if the level statistics is Poisson and random matrix, respectively. Going upwards in the spectrum the transition from Poisson to random matrix statistics is obvious.

The unfolded level spacing statistics between the first and the second, the second and the third, etc., eigenvalues separately was already computed in Ref. [5] and was found to deviate slightly from the Wigner surmise. However, Poisson statistics was not seen there because of the spatial volumes much smaller than ours. Indeed, the spectral density in the Poisson regime is so small that in a small volume even the first few eigenvalues are in the random matrix regime most of the time.

So far we discussed level spacing statistics for a given spatial volume and coarseness of the lattice. For these results to represent real physics it is important to check what happens in the thermodynamic and in the continuum limit. In the remainder of the Letter we address these two questions.

Thermodynamic limit.—We have already seen that the lowest eigenmodes are very localized and their spatial extension is not affected by the volume of the box. Therefore we expect that these modes occur independently and their average number is proportional to the spatial volume. More generally eigenvalues following any intermediate statistics between Poisson and random matrix are also expected to occur in numbers proportional to the spatial volume. If this is true the statistics of eigenmodes number n_1 through m_1 in a spatial volume V_1 should be the same as that of eigenvalues n_2 through m_2 in a spatial volume V_2 provided that $\frac{n_1}{n_2} = \frac{m_1}{m_2} = \frac{V_1}{V_2}$. We verified this by comparing the unfolded level spacing distribution in

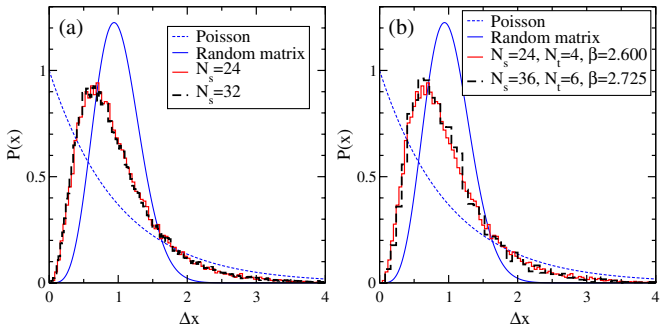

FIG. 4 (color online). The unfolded level spacing distribution for two different spatial box sizes at fixed N_t (a) and for two different N_t's at fixed physical temperature and physical box size (b).

two different spatial volumes, $N_s = 24$ and 32. In Fig. 4(a) we compare the unfolded level spacing distribution of eigenvalues 10–20 in the smaller volume and 24–47 in the bigger volume and find good agreement. To demonstrate that these eigenvalues fall in the intermediate statistics regime we also plot the exponential and Wigner surmise distributions with thin dashed and solid lines. This shows that the density of eigenvalues with a given intermediate statistics scales with the spatial volume, as expected.

Continuum limit.—To see how this picture changes towards the continuum limit we considered an additional ensemble on a finer lattice. The parameters of that, $N_t = 6$, $N_s = 36$, $\beta = 2.725$ were tuned to match the physical temperature and spatial volume of the $24^3 \times 4$ ensemble. Since the average smallest eigenvalue changes roughly with the smallest Matsubara frequency, it does not make sense to compare statistics from the same region in the spectrum of the two ensembles. Instead, we computed the unfolded level spacing distribution of eigenvalues 10–20 on each configuration for both ensembles. As can be seen in Fig. 4(b) there is perfect agreement between the two ensembles.

This means that the deformation of the distribution from Poisson to random matrix statistics occurs through a universal path independently of the lattice spacing. It also implies that the number of very localized Poisson eigenmodes is proportional to the physical spatial volume and not the volume in lattice units. It could suggest that these eigenmodes are localized on some physical gauge field objects that survive the continuum limit with a finite physical density. This scenario is also supported by the fact that the localization range of the eigenmodes in physical units, computed from the quantity $\mathcal{V}_\psi$, was roughly the same on the coarser and finer ensemble.

Discussion.—Low eigenmodes of the QCD Dirac operator are generally explained by mixing instanton and anti-instanton zero modes that could also be natural candidates to explain the localized modes. Assuming independently occurring topological objects, the distribution of the number of zero modes of the overlap Dirac operator on these ensembles can be used to estimate the total density of (anti-)instantons. We found that their density is more than an order of magnitude too small to explain the Poisson modes. Regardless of the origin of the localized modes, the fermion boundary condition clearly plays a crucial role in their appearance because in the opposite $SU(2)$ Polyakov loop sector they are completely absent [15]. We hope to return to a more detailed discussion of the role of the boundary condition in a later publication.

Another important issue is how universal the appearance of Poisson modes is. In Ref. [9] in a range of spatial volumes the lowest two eigenvalues of the overlap Dirac operator were seen to be Poissonian. We extended that study to higher eigenvalues and verified that the overlap spectrum also crosses over from Poisson to random matrix statistics. (Details of this study will appear elsewhere.) This means that the phenomenon we found does not rely on a particular lattice fermion formulation.

This work was supported by OTKA Hungarian Science Fund Grant 49652 and EU Grant (FP7/2007-2013)/ERC No. 208740. We also thank S. D. Katz, D. Nógrádi, and C. Tőke for stimulating discussions.

[1] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[2] J. J. M. Verbaarschot and T. Wettig, Annu. Rev. Nucl. Part. Sci. **50**, 343 (2000).
[3] P. J. Forrester, Nucl. Phys. **B402**, 709 (1993).
[4] F. Farchioni, P. de Forcrand, I. Hip, C. B.Lang, and K. Splittorff, Phys. Rev. D **62**, 014503 (2000).
[5] P. H. Damgaard, U. M. Heller, R. Niclasen, and K. Rummukainen, Nucl. Phys. **B583**, 347 (2000).
[6] R. Pullirsch, K. Rabitsch, T. Wettig, and H. Markum, Phys. Lett. B **427**, 119 (1998).
[7] A. M. Garcia-Garcia and J. C. Osborn, Phys. Rev. D **75**, 034503 (2007).
[8] R. V. Gavai, S. Gupta, and R. Lacaze, Phys. Rev. D **77**, 114506 (2008).
[9] T. G. Kovacs, Phys. Rev. Lett. **104**, 031601 (2010).
[10] P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[11] F. Evers and A. D. Mirlin, Rev. Mod. Phys. **80**, 1355 (2008).
[12] E. Bilgici *et al.*, Few-Body Syst. **47**, 125 (2010).
[13] $\mathcal{V}_\psi$ is simply related to the more commonly used participation ratio (PR). The two differ only in normalization; $\mathrm{PR}_\psi = \mathcal{V}_\psi / V$, where V is the total volume.
[14] M. A. Halasz and J. J. M. Verbaarschot, Phys. Rev. Lett. **74**, 3920 (1995); F. Bruckmann, S. Keppeler, M. Panero, and T. Wettig, Phys. Rev. D **78**, 034503 (2008).
[15] T. G. Kovacs, Proc. Sci. LATTICE2008 (2008) 198.