Evolution and Symmetry of Multipartite Entanglement

Gilad Gour*

*Institute for Quantum Information Science and Department of Mathematics and Statistics, University of Calgary,
2500 University Drive NW, Calgary, Alberta, Canada T2N 1N4*
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We discover a simple factorization law describing how multipartite entanglement of a composite quantum system evolves when one of the subsystems undergoes an arbitrary physical process. This multipartite entanglement decay is determined uniquely by a single factor we call the entanglement resilience factor. Since the entanglement resilience factor is a function of the quantum channel alone, we find that multipartite entanglement evolves in exactly the same way as bipartite (two qudits) entanglement. For the two qubits case, our factorization law reduces to the main result of [T. Konrad, Nature Phys. **4**, 99 (2008)]. In addition, for a permutation P, we provide an operational definition of P asymmetry of entanglement, and find the conditions when a permuted version of a state can be achieved by local means.

With the emergence of quantum information science in recent years, much effort has been given to the study of entanglement [1,2]. It was realized (see [1,2] and references therein) that highly entangled states are the most desirable resources for a variety of quantum information processing tasks, such as quantum teleportation, super-dense coding, entanglement- based quantum cryptography, error correcting codes, and more recently, one-way quantum computation. Because of the effect of decoherence induced by the coupling of the subsystems with the environment, the entanglement of the composite quantum system decreases in time. It is therefore critical, for the implementations of many important quantum information processing tasks, to understand the behavior of entanglement under the influence of decoherence or noise.

To study the evolution of entanglement it seems to be necessary first to study the evolution of the quantum state describing the composite system and then to calculate its entanglement. For example, a situation where no energy is exchanged with the environment, the master equation involving the Lindblad operators can be used to determine the state evolution. Indeed, the elaborate theory on state evolution was the method used by many researchers (e.g., see references in [3]). However, the drawback of this technique is that for multipartite systems (or higher dimensional systems) the state equation can be very hard to solve and therefore the evolution of entanglement can be determined only in very special cases. Quite recently, a new way was found [3] (see also [4] for a similar earlier work) to characterize the evolution of entanglement in two qubits systems, by which the evolution of concurrence [5] (a two qubit measure of entanglement) is determined directly in terms of the evolution of a maximally entangled state, i.e., a Bell state. This technique was generalized to determine the evolution of the G concurrence [6] of two qudits in [7]. In both [3,7] the authors used the Choi-Jamiolkowski isomorphism in order to derive the equations describing

the time evolution of entanglement. Hence, since the Choi-Jamiolkowski isomorphism applies only for bipartite systems, it may give the impression that such entanglement-evolution equations can not be extended to multipartite settings.

In this Letter we discover a simple factorization law describing how multipartite entanglement of a composite quantum system evolves when one of the subsystems undergoes an arbitrary physical process. Quite remarkably, this factorization law holds for an arbitrary number of parties, and reduces to the factorization law given in [3,7] for the bipartite case. Our key idea is to use measures of entanglement that are invariant under the group $G \equiv \mathrm{SL}(d_1, \mathbb{C}) \otimes \mathrm{SL}(d_2, \mathbb{C}) \otimes \cdots \otimes \mathrm{SL}(d_n, \mathbb{C})$, where $d_1, d_2, \ldots, d_n$ are the dimensions of the n subsytems, and $\mathrm{SL}(d, \mathbb{C})$ is the group of $d \times d$ complex matrices with determinant 1. The group G represents (determinant 1) stochastic operations assisted by classical communications (SLOCC) and has been used extensively in the classifications of multipartite entanglement. It is therefore clear from our analysis that even in the bipartite case, it is the invariance under G, rather than the Cho-Jemiolkowski isomorphism, that is necessary for the derivation of the factorization law.

In addition to the factorization law, we also provide an operational definition of P asymmetry of entanglement: a multipartite entangled state contains P asymmetric entanglement if its subsystems can not be permuted (according to the permutation P) by means of LOCC. We show that, in general, states have P-asymmetric entanglement, and by using measures of entanglement that are invariant under G, we are able to generalize the main result of [8] to the case of multipartite systems.

Composite systems of n qubits (with $n > 3$) can be entangled in an uncountable number of ways [9–11] with respect to SLOCC. It is therefore not very clear what role entanglement monotones can play in multiqubits or multiqudits systems unless they are defined operationally.

One exception from this conclusion are entanglement monotones that are defined in terms of SL-invariant polynomials [10–17]. Two important examples are the concurrence [5] and the square root of the three tangle (SRT) [17]. The concurrence and the SRT, respectively, are the only $\mathrm{SL}(2,\mathbb{C}) \otimes \mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SL}(2,\mathbb{C}) \otimes \mathrm{SL}(2,\mathbb{C}) \otimes \mathrm{SL}(2,\mathbb{C})$ invariant measures of entanglement that are homogenous of degree 1. For four qubits or more, the picture is different since there are many homogenous SL-invariant measures of entanglement, such as the square root four tangle [12] or the 24th root of the hyperdeterminant [16]. We now define all such measures that will be satisfying our factorization law; such measures were first discussed in [11].

Definition 1.—Set $\mathcal{H}_n \equiv \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$, and $\mathcal{B}(\mathcal{H}_n)$ the set of all bounded operators (e.g., density matrices) acting on $\mathcal{H}_n$. A SL-invariant multipartite measure of entanglement, E_{inv}, is a nonzero function from $\mathcal{B}(\mathcal{H}_n)$ to the non-negative real numbers satisfying the following: (i) It is SL-invariant; that is, $E_{\mathrm{inv}}(g\rho g^\dagger) = E_{\mathrm{inv}}(\rho)$, for all $g \in G$ and $\rho \in \mathcal{B}(\mathcal{H}_n)$. (ii) It is homogenous of degree 1; i.e., $E_{\mathrm{inv}}(r\rho) = rE_{\mathrm{inv}}(\rho)$ for all nonnegative r and all $\rho \in \mathcal{B}(\mathcal{H}_n)$. (iii) On mixed states it is given in terms of the convex roof extension; that is, $E_{\mathrm{inv}}(\rho) = \min \sum_i p_i E_{\mathrm{inv}}(\psi_i)$, where the minimum is taken with respect to all pure states decompositions of $\rho = \sum_i p_i \psi_i$—here $\psi_i \equiv |\psi_i\rangle\langle\psi_i|$.

Remark.—The criteria in the definition above guarantee that E_{inv} is an entanglement monotone [11]. Note also that the construction via the convex roof extension is consistent with conditions (i) and (ii). The concurrence, the G concurrence, and the SRT are all satisfying the conditions in the definition above. It can be easily checked that E_{inv} is unique (up to multiplication by a positive constant) for the bipartite case with $d_1 = d_2$ and for three qubits, but it is not unique for n-qubits with $n > 3$. Indeed, for four qubits there are four algebraically independent SL-invariant polynomials that generate a whole family of such SL-invariant measures [10,13–15]. Note however that if the dimensions of the subsystems $\{d_i\}$ are not all equal then a SL-invariant measure E_{inv} may not exist. For example, in the bipartite case with $d_1 \neq d_2$, E_{inv} does not exist. For three parties, on the other hand, with $d_1 = d_2 = 2$ and $d_3 = 3$, such a measure exists; it is given in terms of the hyperdeterminant [16].

We now describe briefly the motivation for the criteria in the definition above which is based on some results first discovered in [11] and discussed further in [10]. Let $|\psi\rangle \in \mathcal{H}_n$ and consider the set of (in general non-normalized) states $G|\psi\rangle$ (i.e., the orbit of $|\psi\rangle$ under G). By definition, if $|\psi\rangle$ is generic then $G|\psi\rangle$ is closed. Therefore, for most states $G|\psi\rangle$ is closed. If $G|\psi\rangle$ is not closed then consider its closure, and denote by $|\tilde{\varphi}\rangle$ the state in $G|\psi\rangle$ with the minimum norm; that is $\langle\tilde{\varphi}|\tilde{\varphi}\rangle \leq \langle\tilde{\phi}|\tilde{\phi}\rangle$ for all $|\tilde{\phi}\rangle \in G|\psi\rangle$. The state $|\varphi\rangle \equiv |\tilde{\varphi}\rangle/\sqrt{\langle\tilde{\varphi}|\tilde{\varphi}\rangle}$ is called a *normal form* [11] (see also the critical set in Appendix A of [10]). Moreover,

note that if the normalized state $|\psi\rangle$ is a normal form, then $\| g|\psi\rangle \| \geq 1$ for all $g \in G$, with equality if and only if $g \in \mathrm{SU}(d_1) \otimes \mathrm{SU}(d_2) \otimes \cdots \otimes \mathrm{SU}(d_n)$ [10,11].

The properties (i) and (ii) in the definition above imply (see [15]) that if $|\psi\rangle$ is a normal form, and if $|\varphi\rangle = g|\psi\rangle/\| g|\psi\rangle \|$, then $E_{inv}(|\varphi\rangle) \leq E_{inv}(|\psi\rangle)$. That is, criteria (i) and (ii) imply that among all the states that can be obtained from $|\psi\rangle$ by SLOCC, the normal form $|\psi\rangle$ has the maximum amount of E_{inv}. Indeed, in [10,11] it has been shown that $|\psi\rangle$ is a normal form if and only if each qudit is maximally entangled with the rest of the qudits (i.e., the local density matrices of all qudits are proportional to the identity). Therefore, criteria (i) and (ii) are consistent with this result, and we can consider the normal forms as maximally entangled states. More details and further motivation for the first criterion can be found in the extensive literature on the characterization of entanglement in terms of SL-invariant polynomials (see, for example, [10–16] and references therein).

Lemma 1.—Let $|\psi\rangle \in \mathcal{H}_n$ and E_{inv} as defined in Def. 1. Then, for a matrix $M: \mathcal{H}_n \to \mathcal{H}_n$ of the form $M = A_1 \otimes A_2 \otimes \cdots \otimes A_n$ we have $E_{inv}(M|\psi\rangle\langle\psi|M^\dagger) = 0$ if there exists $1 \leq k \leq n$ such that $\det A_k = 0$.

Proof.—Without loss of generality, assume $\det A_1 = 0$. Denote by $|0\rangle$ a normalized vector in $\mathbb{C}^{d_1}$ such that $A_1|0\rangle = 0$. Denote by $|k\rangle$ (with $k = 1, 2, \ldots, d_1 - 1$) other vectors in $\mathbb{C}^{d_1}$, completing $|0\rangle$ to an orthonormal basis. With this basis we can write

$$M|\psi\rangle = \sum_{k=1}^{d_1-1} |v_k\rangle|\varphi_k\rangle, \quad \text{with} \quad |\varphi_k\rangle \in \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}, \quad (1)$$

and $|v_k\rangle \equiv A_1|k\rangle$ are not necessarily normalized (nor $|\varphi_k\rangle$). Denote by P_r the projection to the space $V = \mathrm{span}\{|v_k\rangle\}_{k=1}^{d_1-1}$, where $r = \dim V$. For $0 < t \in \mathbb{R}$, let D_t be the following $d_1 \times d_1$ matrix: $D_t = t^{r/(r-d_1)}(\mathbb{1} - P_r) + tP_r$. Denote also $g_t \equiv D_t \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$. Clearly, $\det g_t = 1$; i.e., $g_t \in G$. Hence,

$$E_{\mathrm{inv}}[M|\psi\rangle\langle\psi|M^\dagger] = E_{\mathrm{inv}}[g_t M|\psi\rangle\langle\psi|M^\dagger g_t^\dagger]$$
$$= E_{\mathrm{inv}}[t^2 M|\psi\rangle\langle\psi|M^\dagger] = t^2 E_{\mathrm{inv}}[M|\psi\rangle\langle\psi|M^\dagger],$$

where we have used Eq. (1) and criteria (i) and (ii) of Def. 1. Since the above equality is true for all $t > 0$, we must have $E_{\mathrm{inv}}[M|\psi\rangle\langle\psi|M^\dagger] = 0$.

Definition 2.—Let $\$: \mathcal{B}(\mathbb{C}^d) \to \mathcal{B}(\mathbb{C}^d)$, be a quantum channel acting on $d \times d$ positive semidefinite matrices (i.e., density matrices). Any such channel has Kraus representation $\$(\cdot) = \sum_j K_j(\cdot)K_j^\dagger$, with Kraus operators $\sum_j K_j^\dagger K_j \leq \mathbb{1}$. We define the entanglement resilience factor (ERF) of $\$$ to be

$$\mathcal{F}[\$] \equiv \min \sum_j |\det K_j|^{2/d}, \quad (2)$$

where the minimum is taken with respect to all the Kraus representations of $\$$.

Note that $0 \leq \mathcal{F}[\$] \leq 1$ due to the geometric-arithmetic inequality and the fact that for all Kraus representations of \$, $\sum_j K_j^\dagger K_j \leq 1$. Recall also that all Kraus representations of a quantum channel are related by a unitary matrix. In the theorem below we give an operational interpretation for $\mathcal{F}[\$]$.

Theorem 2.—Let $|\psi\rangle \in \mathcal{H}_n$ and $|\phi\rangle \in \mathcal{H}_n$ be two states with nonzero value of E_{inv}. Denote by $\Lambda \equiv \$ \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$, where \$ is an arbitrary quantum channel, which may represent the influence of the environment on the first qudit. Then,

$$\frac{E_{\mathrm{inv}}[\Lambda(|\psi\rangle\langle\psi|)]}{E_{\mathrm{inv}}(|\psi\rangle\langle\psi|)} = \frac{E_{\mathrm{inv}}[\Lambda(|\phi\rangle\langle\phi|)]}{E_{\mathrm{inv}}(|\phi\rangle\langle\phi|)} = \mathcal{F}[\$]. \quad (3)$$

That is, the ratio between the final and initial entanglement depends solely on the ERF of the channel.

Remark.—In the bipartite case with $d_1 = d_2$ the formula above reduces to the one given in [3,7], by replacing the state $|\psi\rangle$ with a maximally entangled state, and by taking E_{inv} to be the concurrence [5] or G concurrence [6] for two qubits or two qudits, respectively. However, in [3,7] the ERF of the channel was not introduced. A remarkable observation is that the ERF depends only on a *single* qudit channel. Now, consider the case of two qubits. Then, by taking E_{inv} to be the concurrence C and replacing $|\psi\rangle$ above with a Bell state $|\psi^+\rangle$ we get the following formula for the ERF:

$$\mathcal{F}[\$] = C[\$ \otimes \mathbb{I}(|\psi^+\rangle\langle\psi^+|)], \quad (4)$$

which can be determined completely by using the Wootters formula. It is remarkable that for any number of qubits and for any choice of E_{inv}, this is the unique formula that is needed to be calculated in order to determine the ERF of a channel acting on a qubit. Similarly, for a channel \$ acting on a qudit, the ERF is given in terms of the G concurrence [6]:

$$\mathcal{F}[\$] = G[\$ \otimes \mathbb{I}(|\psi^+\rangle\langle\psi^+|)], \quad (5)$$

where here $|\psi^+\rangle$ stands for a maximally entangled state in $\mathbb{C}^d \otimes \mathbb{C}^d$. Equation (5) provide the unique value of the ERF and can be used to the determine the ratios in Eq. (3) independent of the choice of E_{inv} or the number of qudits involved.

Proof.—Given the channel $\$(\cdot) = \sum_j K_j(\cdot)K_j^\dagger$, the density matrix $\rho \equiv (\$ \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I})|\psi\rangle\langle\psi|$ has the following pure state decomposition: $\rho = \sum_j |\tilde{v}_j\rangle\langle\tilde{v}_j|$ where $|\tilde{v}_j\rangle \equiv K_j \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}|\psi\rangle$. Note that $|\tilde{v}_j\rangle$ are *not* normalized. Moreover, denote by $\rho = \sum_i |\tilde{w}_i\rangle\langle\tilde{w}_i|$ the optimal decomposition of ρ. That is, denote $p_i \equiv \langle\tilde{w}_i|\tilde{w}_i\rangle$ $E_{\mathrm{inv}}(\rho) = \sum_i E_{\mathrm{inv}}(|\tilde{w}_i\rangle\langle\tilde{w}_i|)$, where we have used the fact that E_{inv} is homogeneous of degree 1. Now, since $\{|\tilde{v}_j\rangle\}$ and $\{|\tilde{w}_i\rangle\}$ are two deferent decompositions of ρ, they are related to each other via a unitary matrix U. That is, if the two sets $\{|\tilde{v}_j\rangle\}$ and $\{|\tilde{w}_i\rangle\}$ do not have the same number of vectors we add zero vectors to the smaller set and then we have

$|\tilde{w}_i\rangle = \sum_j U_{ij}|\tilde{v}_j\rangle = M_i \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}|\psi\rangle$, where U is a unitary matrix, and $M_i \equiv \sum_j U_{ij}K_j$ form another Kraus representation to the *same* quantum channel \$. Now, without loss of generality (see Lemma 1) we can assume that $\det M_i \neq 0$. Hence, we can write

$$|\tilde{w}_i\rangle = (\det M_i)^{1/d}\left[\frac{M_i}{(\det M_i)^{1/d}} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}\right]|\psi\rangle.$$

Since E_{inv} is G invariant and homogeneous, we get $E_{\mathrm{inv}}(|\tilde{w}_i\rangle) = |\det M_i|^{2/d}E_{\mathrm{inv}}(|\psi\rangle)$ and thus

$$E_{\mathrm{inv}}(\rho) = \sum_i |\det M_i|^{2/d}E_{\mathrm{inv}}(|\psi\rangle). \quad (6)$$

What is left to show is that

$$\mathcal{F}[\$] = \sum_i |\det M_i|^{2/d}. \quad (7)$$

To see that, note that the unitary U has been chosen such that the decomposition $\rho = \sum_i |\tilde{w}_i\rangle\langle\tilde{w}_i|$ is optimal. Based on Eq. (6), U has been chosen such that $\sum_i |\det M_i|^{2/d}$ gets the minimum possible value among all the different Kraus representations of \$. Hence, the equality in Eq. (7) must hold.

E_{inv} is a convex function as it is defined in terms of the convex roof extension; hence, we have the following corollary.

Corollary 3.—Let $\rho \in \mathcal{H}_n$ be a multipartite mixed state with nonzero value of E_{inv}, and let Λ and \$ be as in the theorem above. Then, $E_{\mathrm{inv}}[\Lambda(\rho)]/E_{\mathrm{inv}}(\rho) \leq \mathcal{F}[\$]$. The following corollary is an immediate consequence of this equation.

Corollary 4.—Let $\rho \in \mathcal{H}_n$ be a multipartite mixed state with nonzero value of E_{inv}, and let $\{\$_k\}_{k=1,2,\dots,n}$ be a set of n quantum channels. Then,

$$\frac{E_{\mathrm{inv}}[\$_1 \otimes \$_2 \otimes \cdots \otimes \$_n(\rho)]}{E_{\mathrm{inv}}(\rho)} \leq \prod_{k=1}^n \mathcal{F}[\$_k]. \quad (8)$$

As a simple illustration of the above theorem and corollaries, consider the case of three qubits. In three qubits, the only G-invariant measure of entanglement, E_{inv}, is given by the SRT [17] on pure states, and on mixed states it is defined in terms of the convex roof extension. The GHZ state maximizes this measure. Applying the theorem above to this measure gives $E_{\mathrm{inv}}(\$ \otimes \mathbb{I} \otimes \mathbb{I}|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|) = \mathcal{F}[\$]$, where $\mathcal{F}[\$]$ can be calculated via the Wootters formula [see Eq. (4)]. That is, we have found a closed formula for the SRT for all mixed states of the form $\rho = \$ \otimes \mathbb{I} \otimes \mathbb{I}|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$. Moreover, the corollaries above provides upper bounds for E_{inv} on states of the form $\rho = \$_1 \otimes \$_2 \otimes \$_3|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$.

One can also ask how the multipartite entanglement evolve after a separable measurement is performed by the n parties. In the following Lemma we obtain an upper bound on the ratio between the initial entanglement and the final average entanglement after such a measurement.

Lemma 5.—Let $\Lambda(\cdot) = \sum_k M_k(\cdot)M_k^\dagger$ be a trace-preserving separable operation, where $M_k \equiv A_1^{(k)} \otimes A_2^{(k)} \otimes \cdots \otimes A_n^{(k)}$ and $\sum_k M_k^\dagger M_k \leq 1$. Then,

$$\frac{\sum_k p_k E_{\text{inv}}(\sigma_k)}{E_{\text{inv}}(\rho)} \leq \sum_k |\det M_k|^{2/d} \qquad (9)$$

where $\sigma_k \equiv \frac{1}{p_k} M_k \rho M_k^\dagger$ and $p_k \equiv \text{Tr} M_k \rho M_k^\dagger$. Further,

$$\sum_k |\det M_k|^{2/d} = \sum_k |\det A_1^{(k)}|^{2/d} \cdots |\det A_n^{(k)}|^{2/d} \leq 1 \quad (10)$$

with equality if and only if all the operators $\{A_i^{(k)}\}$ are proportional to unitaries. That is, if $\sum_k |\det M_k|^{2/d} = 1$ then Λ is a mixture of product unitary operations.

Proof—The upper bound in Eq. (9) is a direct consequence of conditions (i) and (ii) in Def. 1, and the upper bound in Eq. (10) is a direct consequence of the geometric-arithmetic inequality.

We now show that SL-invariant measures of multipartite entanglement can also be very useful to determine the symmetry of multipartite entanglement.

Definition 3.—Let P be a permutation on n parties. Let $\rho: \mathcal{H}_n \to \mathcal{H}_n$ be a multipartite density matrix and denote by $V_P \rho V_P^\dagger$ the "permuted version" of ρ, where V_P is unitary operator that permutes the subsystems. Then, the entanglement contained in a multipartite state, ρ, is said to be P-symmetric, if by LOCC we can produce the permuted version of the state, i.e. $V_P \rho V_P^\dagger$.

The theorem below generalizes the main result of [8] to multipartite states.

Theorem 6.—Let E_{inv} in Def. 1 be also invariant under some permutation P of the n qudits. Let $\rho: \mathcal{H}_n \to \mathcal{H}_n$ be a mixed state for which $E_{\text{inv}}(\rho) > 0$ and assume that the entanglement of ρ is P symmetric. Then, the permuted version of the state can be achieved by some product unitary operation $U_{A_1} \otimes U_{A_2} \otimes \cdots \otimes U_{A_n}$.

Note, in particular, that if a state with $E_{\text{inv}} > 0$ has different entropies of subsystems, it can not be permuted by LOCC, as local unitaries can not change local entropy. However, if a state has distinct local entropies, but $E_{\text{inv}} = 0$, then it may be possible to permute the state by LOCC. As a simplest example, consider the state $\rho = \rho_{A_1} \otimes \rho_{A_2} \otimes \cdots \otimes \rho_{A_n}$. Clearly, any such state, or its permuted version, can be generated locally even though its local entropies can be distinct. Note also that in three qubits, the SRT is also invariant under permutations [17], and therefore can be used for the theorem above for all permutations P. The same is true for all E_{inv} that are also permutation invariant, such as the four tangle [12] and all hyperdeterminants [16] (see also [14] for other such E_{inv}).

The proof of Theorem 6 is based on Lemma 5 and follows the exact same lines as in Theorem 1 of [8].

In conclusion, Eqs. (3) and (8) provides us, for the first time, with closed expressions for the time evolution of multipartite entanglement of a composite system interacting locally with the environment. These expressions emerge from the SL invariance of the measures defined in Def. 1, and *not* from the Jamiolkowski isomorphism which is the methodology used in Refs. [3,7]. Amazingly, the evolution of multipartite entanglement (under one local channel) is determined completely by the ERF defined in Eq. (2), irrespective to the number of qudits in the system. For multiqubits systems, the ERF has a closed formula given in terms of Wootters concurrence formula. In other words, there is no need to solve any master equations in order to determine the time evolution of multipartite entanglement.

*gour@math.ucalgary.ca
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