Noise-Enhanced Classical and Quantum Capacities in Communication Networks

Filippo Caruso, Susana F. Huelga, and Martin B. Plenio

Institut für Theoretische Physik, Universität Ulm, Albert-Einstein-Allee 11, D-89069 Ulm, Germany

(Received 8 April 2010; published 2 November 2010)

The unavoidable presence of noise is thought to be one of the major problems to solve in order to pave the way for implementing quantum information technologies in realistic physical platforms. However, here we show a clear example in which noise, in terms of dephasing, may enhance the capability of transmitting not only classical but also quantum information, encoded in quantum systems, through communication networks. In particular, we find analytically and numerically the quantum and classical capacities for a large family of quantum channels and show that these information transmission rates can be strongly enhanced by introducing dephasing noise in the complex network dynamics.

DOI: 10.1103/PhysRevLett.105.190501 PACS numbers: 03.67.Hk, 03.65.Yz, 05.40.Ca, 89.75.Hc

Introduction—An important obstacle for the development of quantum communication technologies is the difficulty of transmitting quantum information over noisy quantum communication channels, recovering and refreshing it at the receiver side, and then storing it in a reliable quantum memory [1]. This concerns both point-to-point communication as well as more complex quantum networks consisting of several nodes. These operations are necessary as the unavoidable presence of noise during transmission via a quantum channel and its processing at the receiver's end is generally expected to degrade the transmission quality. It was pointed out, however, that noise may in fact have a positive influence on sustaining quantum correlations [2]. This motivated some early explorations of the potentially beneficial effects that noise may have on information transmission through quantum channels [3]. It was not possible, however, to compute capacities in those examples, and attention focused on related quantities that were furthermore restricted to certain input states. As a result, no firm conclusion could be drawn from these considerations. Recently though, it was realized in a different context that noise may have a positive impact on transport phenomena in complex networks. In fact, it was found that excitation energy transfer (EET) in light harvesting complexes during photosynthesis can benefit considerably from the presence of dephasing noise [4]. Here, it is the intricate interplay of noise and quantum coherence that explains the remarkable efficiency, well above 90%, for EET in light harvesting complexes during photosynthesis, whereas noise-free systems exhibit efficiencies of around 50% only [5]. Motivated by these results, we study the scenario of a realistic communication network, subjected to a noisy evolution, derive analytically and numerically the channel capacities, and we show that they can remarkably increase by using dephasing.

The Model—We consider a generic complex network of N vertices, in which each site represents a two-level quantum system (qubit). Suppose that the sender of the message, Alice (A), wants to transmit a message to the receiver, Bob (B), by using such quantum network [see Fig. 1]. The communication protocol can be the following: (i) Alice applies a swap operation in order to set the unknown initial qubit state ρ_A in the site 1, while the rest of the network is initially prepared in the ground state $|0\dots 0\rangle$; (ii) then they let the network state evolve under some quantum noisy evolution; and (iii) at time $t_{\rm out}$, Bob tries to recover the information sent by Alice through some decoding procedures applied to his output state $\rho_B(t_{\rm out})$, which corresponds to the reduced density operator for the Nth qubit (up to a local unitary transformation). Mathematically, at each time $t_{\rm out}$, one can describe this process as a completely positive and trace-preserving (CPTP) quantum channel of the form

$$\rho_B(t_{\rm out}) \equiv \mathcal{E}(\rho_A) = {\rm Tr}^{(N)}[U(t_{\rm out})(\rho_A \otimes \rho_E)U(t_{\rm out})^\dagger] \quad (1)$$

where ${\rm Tr}^{(N)}$ indicates the trace over all the network qubits but N, ρ_E is the initial state of some effective environment, and $U(t_{\rm out})$ is the unitary evolution of system + environment for a time $t_{\rm out}$. For instance, in the case of a Hamiltonian evolution of the network, ρ_E is the ground state of all the qubits $2, \dots, N$ and $U(t_{\rm out}) = \exp[-iHt_{\rm out}/\hbar]$ is the unitary evolution operator associated to the Hamiltonian H. In this case, following Ref. [6], if the

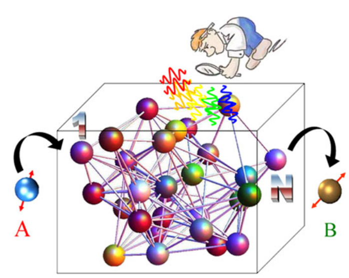

FIG. 1 (color online). Communication network: on one side, Alice sends a message encoded in the qubit 1 while the rest of the network is prepared in its ground state. On the other side, after the network noisy evolution, Bob tries to recover Alice's message decoding the output in site N.

Hamiltonian commutes with the Pauli operator σ_z, i.e., the number of qubits in the state $|1\rangle$ is constant in time, at each time $t_{\rm out}$ the corresponding quantum map is an amplitude damping channel $\mathcal{D}(\eta)$, with the damping coefficient given by $\eta = \langle N|U(t_{\rm out})|1\rangle$, with the convention that $|j\rangle$ denotes the state in which all the qubits are in the state $|\downarrow\rangle$, except the qubit j in the state $|\uparrow\rangle$. In particular, the authors of Ref. [6] considered a spin chain subjected to a Heisenberg Hamiltonian evolution and in this context derived the channel capacities. Here, we investigate a noisy evolution of a network of N qubits, in which, for instance, some pure dephasing noise is present in the dynamics and, as shown later, will play a key role in the information transfer rates of the corresponding communication channel. For simplicity, we will consider the following Hamiltonian, $H = \sum_{j=1}^{N} \hbar\omega_j \sigma_j^+ \sigma_j^- + \sum_{j\neq l} \hbar v_{j,l}(\sigma_j^- \sigma_l^+ + \sigma_j^+ \sigma_l^-)$, and a local Lindblad superoperator that takes into account the dephasing caused by some surrounding environment, i.e., $\mathcal{L}_{\rm deph}(\rho) = \sum_{j=1}^{N} \gamma_j[-\{\sigma_j^+\sigma_j^-, \rho\} + 2\sigma_j^+\sigma_j^-\rho\sigma_j^+\sigma_j^-]$, with σ_j^+ (σ_j^-) being the raising and lowering operators for site j, $\hbar\omega_j$ being the local site excitation energy, $v_{k,l}$ denoting the hopping rate of an excitation between the sites k and l, and γ_j being the dephasing rate at the site j. Generalizing Ref. [6], we find that, at each time $t_{\rm out}$, the corresponding CPTP quantum channel has the form:

$$\mathcal{E}(\eta, s): \rho_A = \begin{pmatrix} p & \delta \\ \delta^* & 1-p \end{pmatrix} \rightarrow \rho_B = \begin{pmatrix} \eta p & \sqrt{\eta s}\,\delta \\ \sqrt{\eta s}\,\delta^* & 1-\eta p \end{pmatrix} \quad (2)$$

with p a real number in the range $[0, 1]$ and δ a complex number such that $|\delta|^2 \leq p(1-p)$. The channel is completely defined by two time-dependent parameters: (i) $\eta \in [0, 1]$ describing the population damping, and (ii) $s \in [0, 1]$ including the dephasing effects. On one hand, because of the linearity of a quantum channel, $\eta(t_{\rm out})$ corresponds to the population of site N at time $t_{\rm out}$ when $\rho_A \otimes \rho_E = |1\rangle\langle 1|$, i.e., one excitation is initially in the site 1. On the other hand, s is generally a complicated time-dependent function of all the parameters involved in the noisy evolution and later will be determined numerically considering a generic input qubit state. Note also that s can be considered as a real number because any phase $e^{i\theta}$ can be eliminated applying a local unitary transformation, i.e., $|1\rangle \rightarrow e^{-i\theta}|1\rangle$, and the quantities analyzed later are invariant under such operations. Besides, it can be easily shown that the map in (2) is equivalent to a consecutive application of an amplitude damping channel $\mathcal{D}(\eta)$ and a phase-flip channel $\mathcal{N}(s)$ (unitarily equivalent to a dephasing channel), changing the phase of the state $|1\rangle$, i.e., $|1\rangle \rightarrow -|1\rangle$, with probability $(1-\sqrt{s})/2$. In other words, one has $\mathcal{E}(\eta, s) = \mathcal{D}(\eta) \circ \mathcal{N}(s) = \mathcal{N}(s) \circ \mathcal{D}(\eta)$. In general, a CPTP quantum channel can be also represented in an elegant form known as operator-sum (or Kraus) representation [1], i.e., $\mathcal{E}(\rho_A) = \sum_k A_k \rho_A A_k^\dagger$, where the so-called Kraus operators A_k satisfy the condition $\sum_k A_k^\dagger A_k = \mathbb{1}$. In particular, these operators for the map in (2) are given by $A_1 = \mathrm{diag}(0, \sqrt{s\eta})$, $A_2 = \mathrm{antidiag}(0, \sqrt{1-\eta})$, and $A_3 = \mathrm{diag}(0, \sqrt{(1-s)\eta})$. In the following, we will analyze the capability of the channel in (2) for transmitting classical and quantum information asymptotically undisturbed, i.e., with a vanishing error probability in the limit of long messages and with some optimal encoding and decoding schemes, by calculating, respectively, the classical and quantum channel capacities.

Classical Capacity—The so-called "one-shot" or product state classical capacity of a quantum channel is obtained maximizing the Holevo information [7], i.e.,

$$C_1(\mathcal{E}) = \max_{\xi_k, \rho_k} \left\{ S\left[\sum_k \xi_k \mathcal{E}(\rho_k)\right] - \sum_k \xi_k S[\mathcal{E}(\rho_k)] \right\} \quad (3)$$

where the maximum is taken over all probability distributions ξ_k and collections of density operators ρ_k, and $S(\rho) = -\mathrm{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy of the state ρ. This is the classical capacity with unentangled encodings (one-shot) while a full maximization over multiple channel uses would provide the unrestricted classical capacity of a quantum channel $C(\mathcal{E})$. For simplicity, we will focus here on the case of unentangled encodings which yields lower bounds for the unrestricted classical capacity of a quantum channel. Notice, however, that the classical capacity with entangled encodings is also a monotonic increasing function of η, by bottleneck inequality, i.e., $C[\mathcal{E}(\eta, s)] \equiv C[\mathcal{E}(\eta', s) \circ \mathcal{E}(\eta/\eta', 1)] \leq \min\{C[\mathcal{E}(\eta/\eta', 1)], C[\mathcal{E}(\eta', s)]\} \leq C[\mathcal{E}(\eta', s)]$ with $\eta \leq \eta'$. Hence, in the presence of a dephasing-induced enhancement of the population transfer η, also the full classical capacity C is enhanced. Now, we calculate C_1 for the family of quantum channels in (2). In the case $s = 1$, the channel reduces to $\mathcal{D}(\eta)$, whose channel capacities were studied in Ref. [6]. Here, first we will solve analytically the case of $s = 0$, which will be relevant in the example below. Let us consider an ensemble $\{\xi_k, \rho_k\}$, with ρ_k of the form of ρ_A in Eq. (2), with parameters p_k and δ_k. The ensemble average state will be a state ρ with coefficients p and δ such that $p = \sum_k \xi_k p_k$ and $\delta = \sum_k \xi_k \delta_k$. In the case of $s = 0$, it can be easily shown that the Holevo information reduces to $H_2(\eta p) - \sum_k \xi_k H_2(\eta p_k)$, with H_2 being the binary entropy function defined as $H_2(x) = -x\log_2 x - (1-x)\log_2(1-x)$. By exploiting the concavity property of H_2, one finds that $C_1[\mathcal{E}(\eta, 0)] \leq \max_{p\in[0,1]}[H_2(\eta p) - pH_2(\eta)]$. The last step of this analytical calculation consists of showing that this upper bound is actually tight and can be obtained with the following optimal ensemble defined by the coefficients $\xi_1 = p$ and generic ξ_i such that $\sum_k \xi_k = 1$, with $p_1 = 1$ and $p_i = 0$, for $i = 2, \ldots d$, where d is the generic dimension of the ensemble $\{\xi_k, \rho_k\}$. Therefore, we find that

$$C_1[\mathcal{E}(\eta, 0)] = H_2(\eta\bar{p}) - \bar{p}H_2(\eta), \quad (4)$$

where $\bar{p} = [(1-\eta)^{(\eta-1)/\eta} + \eta]^{-1}$ is the optimal value of p—see Fig. 2. Notice that, because of the composition law above, one has also that $\mathcal{E}(\eta, s_1) = \mathcal{E}(\eta, s_2) \circ \mathcal{N}(s_1/s_2)$ for $s_1 < s_2$. Hence, by the bottleneck inequality, the following relations hold, i.e. $C_1[\mathcal{E}(\eta, 0)] \leq C_1[\mathcal{E}(\eta, s_1)] \leq \ldots \leq C_1[\mathcal{E}(\eta, s_n)] \leq C_1[\mathcal{E}(\eta, 1)] \equiv C_1[\mathcal{D}(\eta)]$

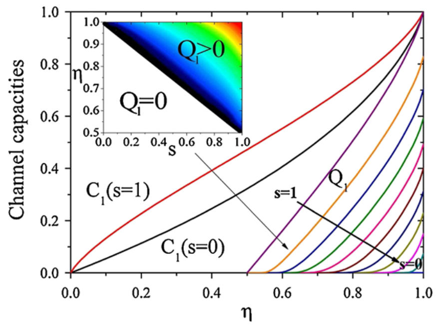

FIG. 2 (color online). Classical and quantum capacities, $C_1[\mathcal{E}(\eta, s)]$ and $Q_1[\mathcal{E}(\eta, s)]$, vs η and s. Inset: contour plot for $Q_1[\mathcal{E}(\eta, s)]$ vs s and η in a color gradient scheme where light grey corresponds to 1 and black to vanishing values of Q_1. Note that Q is always zero for $\eta \leq 1/2$ and any s.

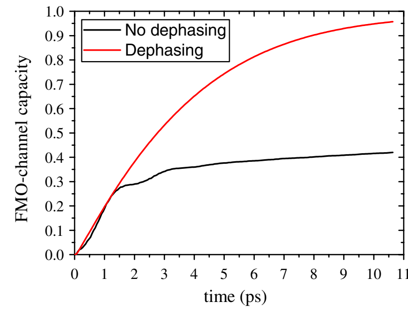

FIG. 3 (color online). Classical capacity of the FMO-complex quantum channel, $C_1\{\mathcal{E}[p_{\rm sink}(t_{\rm out}), 0]\}$, vs time $t_{\rm out}$. There is a clear remarkable enhancement of C_1 in the case of pure dephasing noise. By maximizing C_1 over $t_{\rm out}$, one has $\bar{C}_1(\mathcal{E}) \sim 0.72$ in absence of dephasing, while $\bar{C}_1(\mathcal{E}) \sim 0.99$ in the case of dephasing.

for $0 \leq s_1 \leq \ldots \leq s_n \leq 1$. In other words, the η dependence of $C_1[\mathcal{E}(\eta, s)]$ will be described by a continuous family of lines for the intermediate value of s in the range $[0, 1]$, i.e., between the red ($s = 1$) and black ($s = 0$) extreme lines in Fig. 2. From these results it turns out that the presence of dephasing, on one hand, may increase the value of η enhancing also C_1, but, on the other hand, may reduce the value of C_1 by decreasing s. Hence, a compromise of these two dephasing-induced effects could lead to a global enhancement of the classical capacity, as shown below. Since the classical capacity does implicitly depend on the reading time $t_{\rm out}$, we define also a new quantity, $\bar{C}_1(\mathcal{E})$, as the *best* channel capacity obtained maximizing the capacity over $t_{\rm out}$, i.e., $\bar{C}_1(\mathcal{E}) = \max_{t_{\rm out}} C_1\{\mathcal{E}[\eta(t_{\rm out}), s(t_{\rm out})]\}$.

As a concrete example, we will consider the transport dynamics of electronic excitations in a biological pigment-protein complex, called the FMO (Fenna-Matthews-Olson) complex, involved in the early steps of photosynthesis in sulfur bacteria [8]. It is possible to describe this complex as a network of seven sites, represented as qubits, and the excitation transport by the Hamiltonian and Lindblad superoperators as above—see Ref. [5] for details. At each time $t_{\rm out}$ (scale of picoseconds), the transfer of energy from site 1 to the so-called reaction center (RC) can be mapped as in Eq. (2), where η is equal to the amount of excitation $p_{\rm sink}(t_{\rm out})$ in the RC when one excitation is initially in site 1, while $s = 0$ since there is just an irreversible population transfer from the site 3 to the RC, i.e., the map is $\mathcal{E}[p_{\rm sink}(t_{\rm out}), 0]$. By choosing the parameters as in Ref. [5], we find that the classical capacity of the FMO complex, described as a quantum channel, $C_1\{\mathcal{E}[p_{\rm sink}(t_{\rm out}), 0]\}$, is remarkably enhanced in the presence of dephasing, especially after 1 ps–see Fig. 3. The dephasing-enhanced classical capacity is due to the acceleration of transport in the network and may have been expected from the results of [4,5] as dephasing does not affect classical information. Dephasing, however, destroys quantum information and so it is unexpected that the quantum capacity may be enhanced by dephasing as well.

Quantum capacity—The quantum capacity Q refers, instead, to the coherent transmission of quantum information (measured in number of qubits) through a quantum channel. It is more difficult to treat than the classical capacity discussed above and its explicit calculation is one of the basic issues in quantum information science. The one-shot formula for the quantum capacity is obtained maximizing the coherent information [9], i.e.,

$$Q_1(\mathcal{E}) = \max_{\rho \in \mathcal{H}} \{S[\mathcal{E}(\rho)] - S(\rho, \mathcal{E})\} \qquad (5)$$

where the maximization is performed over all qubit states in the input Hilbert space $\mathcal{H}$. Here, $S(\rho, \mathcal{E})$ is the exchange entropy of the channel [1], describing the amount of information exchanged between the system and the environment after the noisy evolution, and is given by $S(\rho, \mathcal{E}) \equiv S(W) = -{\rm Tr}[W \log_2 W]$ with $W_{ij} = {\rm Tr}[A_i \rho A_j^\dagger]$. Note that Q_1 is usually a lower bound for Q (which is maximized over many channel uses), since the coherent information is generally not additive. It turns out numerically that the expression to maximize in (5), with ρ as ρ_A in (2), is always decreasing in $|\delta|^2$ for $\eta \geq 1/2$ and then it achieves the maximum value for $\delta = 0$; the remaining optimization in p has been performed numerically and the results are shown in Fig. 2. In the case of $s = 1$, the channel reduces to $\mathcal{D}(\eta)$, for which the coherent information can be proved to be additive (i.e., since it is a degradable channel [10]), and the optimization over the channel uses is not necessary; this regime has been investigated in Ref. [6]. Here, we generalize those results in the presence of dephasing, i.e., $s < 1$, and find numerically the one-shot quantum capacity Q_1 as in Fig. 2. The additivity of the coherent information cannot be proved for $s < 1$ but our results for Q_1 are, of course, a lower bound for the capacity Q with entangled encodings. Although we have not computed the full quantum capacity rigorously and analytically, our lower bounds are sufficient to show the noise-assisted enhancement for the unrestricted quantum capacity as well. Similar enhancement is observed in other figures of merit, i.e., channel fidelity $F(R)$ and entropy $S(R)$ [11,12], which can be analytically derived for the map in (2), i.e., $F(R) = 1/4(1 + \eta + 2\sqrt{\eta s})$ and

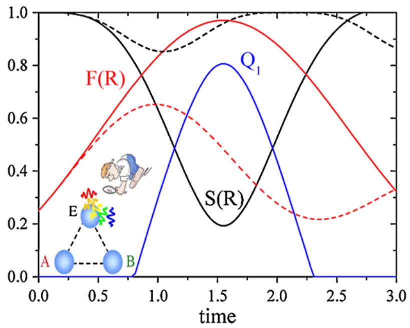

FIG. 4 (color online). Quantum capacity Q_1 for a three-site network vs time t_{out}. In the absence of noise, Q is always zero, while Q_1 can get values almost up to one in the presence of dephasing. Here, the hopping and dephasing rates are $v_{12} = v_{23} = v_{13} = 1$, and $\gamma_2 = 50$, $\gamma_1 = \gamma_3 = 0$. The corresponding channel entropy and fidelity are also shown in the noiseless (dashed) and dephasing (continuous line) case. In particular, $\bar{Q}_1(\mathcal{E}) = 0$ without noise, while $\bar{Q}_1(\mathcal{E}) \sim 0.8$ with $\gamma_2 = 50$ and is monotonically increasing to 1 for higher dephasing rates.

$S(R) = -\sum_{i=1}^{3} \lambda_i \log_2 \lambda_i$, with $\lambda_1 = (1-\eta)/2$, $\lambda_{2,3} = 1/4[1 + \eta \pm \sqrt{4\eta s + (1-\eta)^2}]$. Because of the composition law $\mathcal{E}(\eta, s) = \mathcal{D}(\eta) \circ \mathcal{N}(s)$, the channel in (2) has always a vanishing Q for $\eta \leq 1/2$ and for any s, since in this regime $\mathcal{D}(\eta)$ has $Q = 0$ (being an antidegradable channel) [6,13]. Finally, as in the classical capacity case, we introduce the quantity $\bar{Q}_1(\mathcal{E}) = \max_{t_{\text{out}}} Q_1\{\mathcal{E}[\eta(t_{\text{out}}), s(t_{\text{out}})]\}$. A specific example of a three-qubit network is shown in Fig. 4. The presence of dephasing ''switches on'' the channel capability of transmitting quantum information and the optimal rates can be very close to 1 (i.e., almost perfect state transfer), while Q is exactly zero without dephasing. The intuitive reason for this behavior is the fact that, in the noise-free case, quantum information progresses along two possible paths, thus being split and not arriving at the same time. This approximates a channel that splits the quantum information which in turn has vanishing quantum capacity. For strong dephasing, the path via E is blocked and direct transfer from A to B leads to the arrival of all quantum information at B. Noise-assisted channel capacities may also be observed for larger networks and when all sites suffer dephasing, but the effect is most pronounced for nonuniform distribution of the noise. As final remark, notice that, in a quantum cryptographic scenario, the dephasing can be induced by the presence of an eavesdropper (Eve) in the third site, and, interestingly enough, it turns out that the eavesdropping operation is completely useless for Eve (i.e., Q for the channel $A \rightarrow E$ remains exactly zero), but it does sensibly improve the Alice-Bob communication.

Conclusions and Outlook—We have evaluated analytically and numerically the classical and quantum channel capacities of a realistic communication network and showed that these optimal information transmission rates can be enhanced by applying some pure dephasing to the network. In particular, this allows us to reinterpret the observed dephasing-assisted EET in photosynthetic complexes as an example of a quantum channel with a noise-enhanced classical capacity. Perhaps more surprisingly, we have shown that the presence of noise may lead to a finite quantum capacity where the noiseless system has vanishing capacity. As a result, not only the transmission rate of classical information can be assisted by noise but also the transmission of quantum information coded in quantum states. We expect these results to be easily generalizable to bosonic systems and to be valid for any Hamiltonian preserving the number of excitations, other forms of noise, and also for non-Markovian evolutions. Finally, the three-site quantum network illustrating the fundamentals of our results could be experimentally investigated relatively easily by considering, e.g., quantum information platforms using trapped ions or cold atoms where dephasing can be introduced in a controlled manner.

This work was supported by EPSRC Grant No. EP/C546237/1, the EU STREP project CORNER, the EU Integrated project QAP, the Royal Society, a Marie Curie Intra European Action, and the Humboldt Foundation.

[1] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
[2] M. B. Plenio and S. F. Huelga, Phys. Rev. Lett. **88**, 197901 (2002); L. Hartmann, W. Dur, and H. J. Briegel, Phys. Rev. A **74**, 052304 (2006).
[3] C. Di Franco *et al.*, Phys. Rev. A **77**, 062337 (2008); G. Bowen and S. Mancini, Phys. Lett. A **321**, 1 (2004); **352**, 272 (2006).
[4] M. Mohseni *et al.*, J. Chem. Phys. **129**, 174106 (2008); M. B. Plenio and S. F. Huelga, New J. Phys. **10**, 113019 (2008).
[5] F. Caruso *et al.*, J. Chem. Phys. **131**, 105106 (2009); Phys. Rev. A **81**, 062346 (2010); P. Rebentrost *et al.*, New J. Phys. **11**, 033003 (2009); A. W. Chin *et al.*, New J. Phys. **12**, 065002 (2010).
[6] V. Giovannetti and R. Fazio, Phys. Rev. A **71**, 032314 (2005).
[7] A. S. Holevo, IEEE Trans. Inf. Theory **44**, 269 (1998); P. Hausladen *et al.*, Phys. Rev. A **54**, 1869 (1996); B. Schumacher and M. D. Westmoreland, *ibid.* **56**, 131 (1997).
[8] H. Lee *et al.*, Science **316**, 1462 (2007); V. I. Prokhorenko *et al.*, J. Phys. Chem. B **106**, 9923 (2002); J. Adolphs and T. Renger, Biophys. J. **91**, 2778 (2006).
[9] S. Lloyd, Phys. Rev. A **55**, 1613 (1997); H. Barnum, M. A. Nielsen, and B. Schumacher, Phys. Rev. A **57**, 4153 (1998); I. Devetak, IEEE Trans. Inf. Theory **51**, 44 (2005).
[10] I. Devetak and P. W. Shor, Commun. Math. Phys. **256**, 287 (2005).
[11] W. Roga *et al.*, J. Phys. A **41**, 035305 (2008).
[12] D. Kretschmann and R. F. Werner, New J. Phys. **6**, 26 (2004).
[13] F. Caruso and V. Giovannetti, Phys. Rev. A **74**, 062307 (2006); Phys. Rev. A **76**, 042331 (2007).