Self-Similar Modes of Coherent Diffusion

O. Firstenberg,[1] P. London,[1] D. Yankelev,[1] R. Pugatch,[2] M. Shuker,[1] and N. Davidson[2]
[1]Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel
[2]Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
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Self-similar solutions of the coherent diffusion equation are derived and measured. The set of real similarity solutions is generalized by the introduction of a nonuniform phase, based on the elegant Gaussian modes of optical diffraction. In a light-storage experiment, the complex solutions are imprinted on a gas of diffusing atoms, and the self-similar evolution of both their amplitude and phase pattern is demonstrated. An algebraic decay depending on the mode order is measured. Notably, as opposed to the regular diffusion spreading, a subset of the solutions exhibits a self-similar contraction.

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The diffusion of a vector field delineated in the Torrey-Bloch formulation [1] dictates the coherent dynamics of various spin ensembles, including the nuclear magnetization in NMR [1,2], spin-polarized alkalis in vapor [3,4], and electronic or exciton spins in spintronics systems, namely, metals, ferromagnets, and semiconductors [5–8]. In their most commonly used form, the Torrey-Bloch equations are reduced into a single diffusion equation,

$$\frac{\partial}{\partial t}\psi(\mathbf{r},t) = D\nabla^2\psi(\mathbf{r},t) - \gamma\psi(\mathbf{r},t), \tag{1}$$

where D is the diffusion constant and $\gamma = 1/T_2$ is the homogenous decay rate. The field ψ is complex valued when the dynamics of the (transverse) spin coherence is considered [1], or real-valued for a (longitudinal) spin-population dynamics [9]. In the former, phase diffusion is of major importance. Alternatively, light-matter excitations in these media, such as dark-state polaritons and exciton polaritons, exhibit coherent diffusion in the form of Eq. (1), due to the ''matter-part'' contribution to their dispersion relation [10]. In fact, vorticity [11], atomic (nonspin) coherence [4,12], and even temperature [13] are all described by Eq. (1) when subjected to diffusion.

In this Letter, we study the self-similarity properties of coherent diffusion in one or two dimensions. Self-similarity is generally associated with the long-time behavior of dynamic processes [14]. In most dissipative systems, similarity solutions decay with a characteristic rate, indicating the asymptotic evolution of a given initial condition [15,16]. Similarity solutions emerge also in non-dissipative systems and often prevail, with a familiar example being the family of Gaussian beams in free-space paraxial optics [17], when the propagation distance is given the role of time. These beams are broadly referred to as *modes* of diffraction, even though they are not stationary as Bessel beams [18] and not eigenmodes of the underlying Hamiltonian.

For coherent diffusion, stationary modes were extensively employed [1,9], while self-similar modes would have been much more natural in unbounded domains. These modes can also be utilized for the construction and analysis of stable evolution in coherent systems. Here, we present the complex self-similar solutions and experimentally follow their dynamics, utilizing direct observation and spatial mapping techniques unavailable in NMR or spintronics systems. We demonstrate the preservation of both their shape and phase pattern, as well as their characteristic decay. Similarly to optical modes, the diffusion modes are only partially self-similar; their shape is preserved up to scaling and normalization, while their phase pattern curves or flattens. A self-similar contraction, resembling focus or collapse [19], is also demonstrated.

Consider first the time-independent paraxial diffraction, $\partial E/\partial z = -i\nabla_\perp^2 E/(2k)$, for the slowly-varying envelope, E, of a light field with a wave number k. Two different sets of polynomial-Gaussian solutions are known for this equation, namely, the standard and the elegant beams [17]. The more familiar ''standard'' modes, e.g., the Hermite Gaussian (HG) or Laguerre Gaussian (LG), form complete sets of modes that are self-similar under diffraction. Their transverse intensity distribution, $I(x, y; z)$, is maintained along the propagation direction z, normalized, and scaled by the beam radius $w(z)$. In contrast, the transverse shape of the ''elegant'' solutions is generally not maintained, and originally they were investigated due to their elegant mathematical form [20,21].

The elegant HG solution, $E^{\mathrm{HG}}_{n_1,n_2}(\mathbf{r}; w_0)$, with w_0 being the radius at the waist plane $z = 0$, is written in terms of the Hermite polynomials of orders n_1 and n_2 as,

$$E^{\mathrm{HG}}_{n_1,n_2} = \frac{E_0}{kw_0}\left[\frac{kw_0^2}{2q(z;w_0)}\right]^{N/2+1} H_{n_1}(\tilde{x})H_{n_2}(\tilde{y})e^{-\tilde{x}^2-\tilde{y}^2}. \tag{2}$$

Here, $q(z; w_0) = iz_R + z$ is the complex radius, $z_R = kw_0^2/2$ is the Rayleigh length, $N = n_1 + n_2$ is the total mode order, and E_0 is a normalization constant. The transverse scaling, appearing in both the polynomial and the Gaussian terms, depends on z and w_0, with $\tilde{x} = x[ik/2/q(z;w_0)]^{1/2}$ and $\tilde{y} = y[ik/2/q(z;w_0)]^{1/2}$.

The beam radius, $w(z)$, and the radius of curvature of the phase fronts, $R(z)$, are obtained from $q(z;w_0)^{-1} = R(z)^{-1} - (i2/k)w(z)^{-2}$. For the corresponding standard mode, the complex arguments of the polynomial $(\tilde{x}, \tilde{y})$ are replaced by real arguments $[\sqrt{2}x/w(z), \sqrt{2}y/w(z)]$. Thus "elegant" modes with a homogeneous polynomial are also "standard," and we denote them as "common" modes.

Suppose that diffusion takes place at the (x, y) plane, while z is constant. Equation (2), with $\partial E/\partial z = -i\nabla_\perp^2 E/(2k)$ and $(\partial/\partial w_0)_z = ikw_0(\partial/\partial q)_{w_0} + (\partial/\partial w_0)_q$, then gives

$$\nabla_\perp^2 E^{\mathrm{HG}}_{n_1,n_2} = \frac{2}{w_0}\left(\frac{\partial E^{\mathrm{HG}}_{n_1,n_2}}{\partial w_0}\right)_z - \frac{2}{w_0^2}(N+1)E^{\mathrm{HG}}_{n_1,n_2}. \quad (3)$$

Under diffusion, the first term accounts for a change in the waist radius, and the second term for a uniform decay of the field. Equation (1) is therefore solved by

$$\psi^{\mathrm{HG},z}_{n_1,n_2}(x,y,t) = e^{-\gamma t}s(t)^{-(N+1)}E^{\mathrm{HG}}_{n_1,n_2}[\mathbf{r}; w_0 s(t)], \quad (4)$$

where the diffusion coefficient enters only through the waist stretching factor, $s(t) = (1 + 4Dt/w_0^2)^{1/2}$. Thus the spatial consequence of diffusion is an effective stretching of the beam radius at the waist plane, even when the diffusion occurs far from the waist ($z \neq 0$), as illustrated in Fig. 1. Note that if diffusion is addressed as an imaginary-time evolution of diffraction, it is readily seen from the definition of the complex radius $q(z;w_0)$ that exchanging the real evolution in z for an imaginary one corresponds to a real increase in the waist radius. The total power, $P(t) = \int dxdy|\psi^{\mathrm{HG},z}_{n_1,n_2}|^2$, which is independent of z, is not preserved under diffusion even when $\gamma = 0$, due to an algebraic decay term $s(t)^{-(N+1)}$. This occurs even for the lowest Gaussian mode ($N = 0$) because it is the field, rather than the intensity, that is diffusing. Higher-order modes ($N > 0$) decay faster due to the diffusion of the nonhomogenous phase pattern, which contains larger gradients for higher N. A similar procedure can be carried out for an elegant LG solution, $E^{\mathrm{LG}}_{p,m}$, of radial order p and orbital order m, yielding Eq. (4) for the diffusing field $\psi^{\mathrm{LG},z}_{p,m}(x,y,t)$, with $N = p + m$.

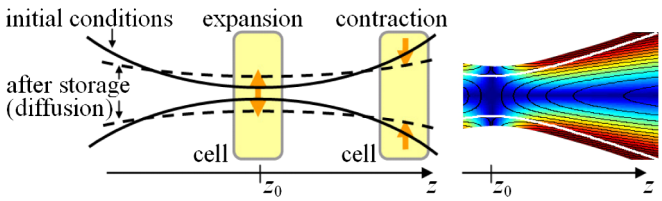

FIG. 1 (color online). The effect of diffusion on Gaussian beams is an effective expansion of the waist, even if it occurs away from the waist plane. Therefore far enough from the waist, the diffusion contracts the transverse shape. Right (color) map: Phase gradients of the field envelope (black lines are equal phase contours; white lines are the beam outline). From the viewpoint of the microscopic atomic motion, the contraction far from the waist occurs due to destructive interference of atoms diffusing through the rapidly oscillating phase pattern (red colored).

At the waist, all arguments in Eq. (2) are real, and the HG solutions $\psi^{\mathrm{HG},z=0}_{n_1,n_2}$ are identified with the expanding similarity solutions of real-valued diffusion [15], occurring, e.g., for the vorticity field of a viscous fluid [11,16]. The real solutions are alternatively derived from a given self-similar solution with a single real scaling $\{\psi(\mathbf{r},t) = h(t)f[\mathbf{r}/w(t)]\}$, by taking any of its spatial derivatives $\{\partial^n\psi/\partial x^n = h(t)f^{(n)}[\mathbf{r}/w(t)]/w(t)^n\}$. Indeed, the derivatives of the lowest-order Gaussian beam constitute the elegant modes, with the derivative order corresponding to the total mode order N, and with the possible generalization for unified Hermite-Laguerre-Gaussian modes [22,23]. Consequently, $\psi^{\mathrm{HG},z=0}_{n_1,n_2}$ or $\psi^{\mathrm{LG},z=0}_{p,m}$, and all linear combinations of them with the same total order, are self-similar, sharing the same stretching and the same algebraic decay. Alternatively, any complex "image" in the (x, y) plane can be expanded in terms of $\psi^{\mathrm{HG},z=0}_{n_1,n_2}$ or $\psi^{\mathrm{LG},z=0}_{p,m}$ using their biorthogonal pairs [24], its diffusion can be described in terms of the modes dynamics, and asymptotically the lowest-order solution prevails [25].

To validate the above predictions, we perform experiments with thermal alkali atoms confined in a vapor cell with a buffer gas. This system attracted considerable recent study, exhibiting spectral fringes, narrowing, and coherent recurrence [12,26,27], slow-light manipulation [28], and spatial diffusion [4,29–31]. Using the technique of light storage and retrieval [32], any arbitrary initial condition can be imprinted on the diffusing atoms, and the subsequent dynamics can be observed [4]. A resonant laser beam with the desired optical mode is sent into the cell, and its complex field envelope is mapped onto the atomic coherence field by shutting down an auxiliary control beam. The coherence field, ψ, is allowed to evolve for a controlled duration τ, in which the alkali diffusion through the buffer gas takes place, as well as a homogenous decoherence γ (e.g., due to spin-exchange relaxation [33]). The coherence is then converted back to light, which is imaged onto a camera. The setup is similar to that described in Ref. [29], where the topological stability of the stored vortex $E^{\mathrm{LG}}_{0,1}$ was confirmed.

The experiment was carried out with the fundamental Gaussian mode $E^{(\mathrm{HG/LG})}_{0,0}$, the LG modes $E^{\mathrm{LG}}_{0,1}$ and $E^{\mathrm{LG}}_{0,2}$, and the HG mode $E^{\mathrm{HG}}_{0,1}$. Figure 2(a) presents the retrieved images, proportional to $|\psi(x,y,\tau)|^2$ for a storage performed with the cell located at the beam waist ($z = 0$), for durations of $\tau = 2$, 30, and 60 μs. Evidently, all the modes expand but maintain their shape through the diffusion process. As a complementary test, we have also passed the retrieved beams through a binary grating mask with a fork dislocation, which adds a phase function $m\phi$ in its mth diffraction order [34]. After the mask, as shown in Fig. 2(b), the retrieved vortex mode $E^{\mathrm{LG}}_{0,1}$ ($m = 1$) produces a Gaussian ($m = 0$) and a higher-order vortex ($m = 2$) in the -1 and $+1$ diffraction orders, confirming the

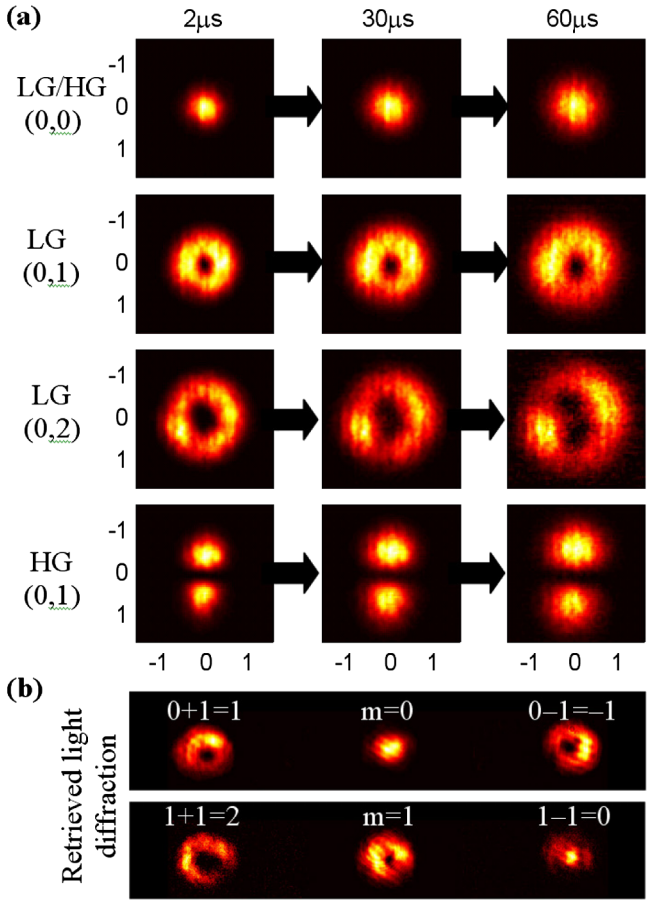

FIG. 2 (color online). (a) Self-similar diffusion of the complex field of atomic coherence in a light-storage experiment. Top to bottom: the basic Gaussian mode, Laguerre-Gaussian modes of radial order $p = 0$ and orbital orders $m = 1, 2$, and a Hermite-Gaussian mode of Cartesian orders (0, 1). All images are 1.6×1.6 mm. (b) Images after storage, diffracted by a binary grating mask with a fork dislocation, for confirming the conservation of phase. The retrieved Gaussian mode (top) yields the two $m = \pm 1$ vortex modes, while the retrieved vortex $m = +1$ (bottom) produces an $m = 0$ and an $m = +2$ modes.

maintenance of the phase pattern. Figure 3(a) presents the increase in the waist-radii squared versus the storage duration, showing the same linear increase for all curves, $w(\tau)^2 - w_0^2 = w_0^2[s(\tau)^2 - 1] = 4D\tau$. The cross sections shown in Fig. 3(b), scaled according to $s(\tau)$ and normalized, clearly demonstrate the self-similarity. The algebraic decay of the diffusing modes, $s(t)^{-(N+1)}$, is measured by integrating over the intensity of the retrieved images [Fig. 4]. All modes exhibit a significant algebraic decay on top of the homogenous decay, with the higher-order modes decaying faster, showing an excellent quantitative agreement with the predictions of Eq. (4).

We now discuss the diffusion of the elegant modes at a given plane outside the waist plane, $z \neq 0$. In optics, an expansion of the waist results in an increase of the beam's transverse size for $|z| < z_R$ and in a decrease for $|z| > z_R$ [Fig. 1]. It follows, perhaps counterintuitively, that the initial effect of diffusion occurring at $|z| > z_R$ is a contraction.

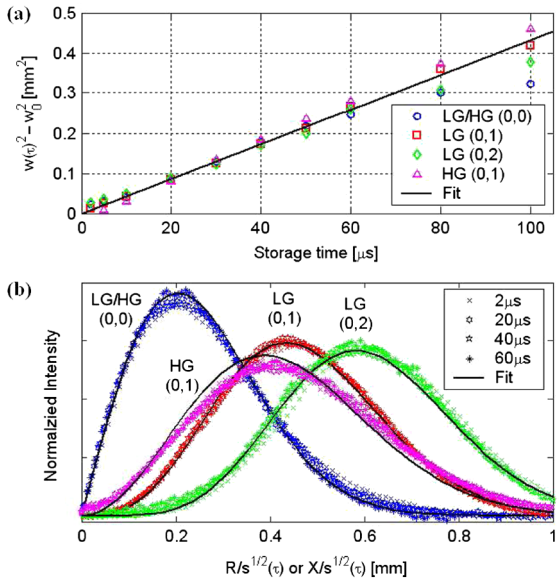

FIG. 3 (color online). (a) Linear growth of w^2 with respect to the time duration of diffusion. The line is $w^2 - w_0^2 = 4D\tau$, with D and w_0 fit parameters ($w_0 = 0.4$–0.55 mm, varying for the different modes, and $D = 10.8$ cm^2/s). (b) Self-similarity: cross sections (Cartesian or radially weighted) at different times are congruent when the coordinate is scaled with $s(\tau)^{-1/2}$. The solid lines are the exact elegant HG and LG modes.

Locally, it is the consequence of a destructive interference in regions of the beam where the phase pattern rapidly oscillates [Fig. 1, right]. The contraction versus diffusion time, $C(\tau) = w(\tau)/w(0)$, is given by $C(\tau)^2 = [s(t)^4 + \rho^2]/[s(t)^2(1 + \rho^2)]$, where $\rho = z/z_R$ specifies the initial distance from the waist. As the waist radius increases

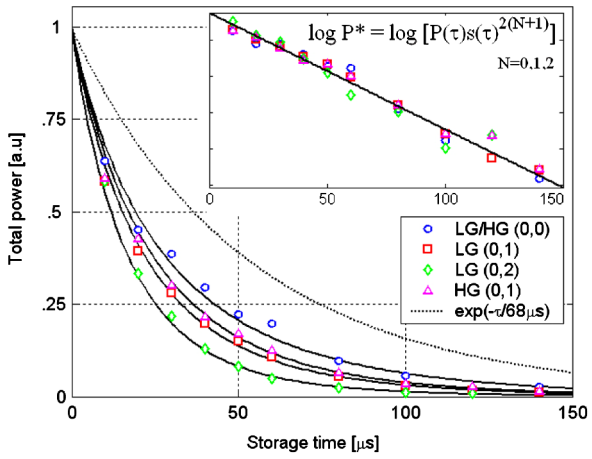

FIG. 4 (color online). Decay of the total power in the retrieved images $P(\tau) = \int dxdy|\psi(\tau)|^2$. In the inset, P^* compensates for the algebraic decay [$s(\tau)$ known from Fig. 3] and collapses onto a single straight line in the semilog scale, yielding the homogenous decay rate $2\gamma = (68\ \mu s)^{-1}$. In the main graph, the dashed line is $e^{-2\gamma\tau}$ and the solid lines are $e^{-2\gamma\tau}s(\tau)^{-2(N+1)}$, demonstrating the faster decay of the higher-order modes. The difference between the $LG_{0,1}$ and the $HG_{0,1}$, both having $N = 1$, is due to slightly different initial waist radii (w_0).

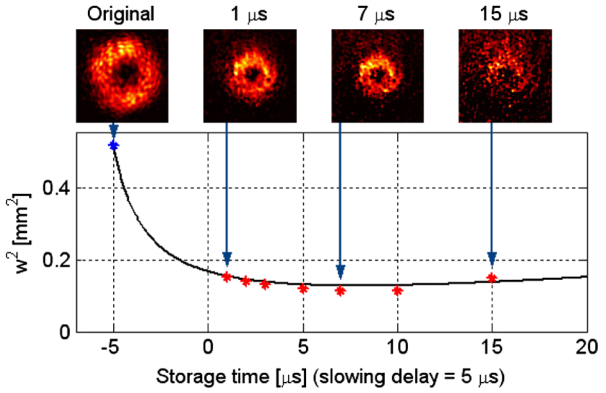

FIG. 5 (color online). The self-similar contraction and subsequent expansion of the $LG_{0,1}$ mode upon diffusion. Here, we refer to the original beam as the initial condition (left, taken off resonantly), since much contraction occurs already during the slow-light propagation. The original mode has a radius of 700 μm and a phase curvature of $(250\ \mathrm{mm})^{-1}$. The minimal radius, of 340 μm, is obtained after slowing for 5 μs plus storage for 7 μs. The black line is calculated from $C(\tau)^2$ with no fit parameters. In these conditions, the power decays substantially faster than in the expansion experiments, and substantial noise is already apparent after 15 μs of storage.

during diffusion, $z_R(\tau)$ increases and eventually crosses the observed plane (which z coordinate is constant). At this time, the maximal contraction $C^2_{\mathrm{min}} = 2\rho/(1+\rho^2)$ is obtained, and thenceforth the beam expands indefinitely.

Elegant modes, as opposed to the standard modes, are generally not self-similar under *diffraction*, and the shape of the beam depends on z/z_R. Hence, even when z is held constant, the increase of z_R during *diffusion* changes the transverse shape and breaks its self-similarity. However, the aforementioned "common" modes, which are simultaneously elegant and standard, are self-similar under diffraction and thus also self-similar under diffusion even for $z \neq 0$. The HG modes of orders 0 and 1 [17] and all LG modes with $p = 0$ (the vortex modes) are such common modes. Far from the waist, at $z > z_R$, these modes contract self-similarly. Figure 5 presents the experimental result for a diffusing $E^{\mathrm{LG}}_{0,1}$ mode focused at a distance $8z_R$ before the cell ($\rho = 8$), yielding $C^2_{\mathrm{min}} \approx 1/4$.

Finally, we point out an intriguing instability phenomenon noticeable in Fig. 2(a) for the $E^{\mathrm{LG}}_{0,2}$ mode. During diffusion, the $m = 2$ vortex breaks down into two vortices, probably of $m = 1$. A decay of high-order vortices into lower-order ones has been seen also in optics, quantum fluids, and Bose-Einstein condensates. Here, several candidate mechanisms may be responsible for the imperfection [11,34], which evidently conserves the cross section of the original vortex [Fig. 3(b)].

In conclusion, the self-similar evolution of coherent diffusion can be inferred from the expansion of elegant optical modes at their waist, which may also lead to a self-similar contraction. An algebraic decay of the power occurs even for the lowest-order mode due to field interference effects. The elegant and standard modes form complete sets of similarity solutions for diffusion and diffraction, respectively, while the simultaneous process of diffusion and diffraction, as occurring for light-matter polaritons [28], is yet to be explored. Moreover, the self-similar modes can be readily applied in NMR and spintronics systems governed by transverse spin dynamics, while extended *vector* solutions should be obtained when the dynamics of the full vector field is of significance.

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