Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories

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We find a surprising connection between asymptotically flat spacetimes and nonrelativistic conformal systems in one lower dimension. The Bondi-Metzner-Sachs (BMS) group is the group of asymptotic isometries of flat Minkowski space at null infinity. This is known to be infinite dimensional in three and four dimensions. We show that the BMS algebra in 3 dimensions is the same as the 2D Galilean conformal algebra (GCA) which is of relevance to nonrelativistic conformal symmetries. We further justify our proposal by looking at a Penrose limit on a radially infalling null ray inspired by nonrelativistic scaling and obtain a flat metric. The BMS_4 algebra is also discussed and found to be the same as another class of GCA, called semi-GCA, in three dimensions. We propose a general BMS-GCA correspondence. Some consequences are discussed.

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Introduction.—Holography in asymptotically anti–de Sitter spaces has been the cynosure of attention for over a decade, following the anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1]. Somewhat less studied and even lesser understood is holography in asymptotically flat spacetimes [2]. One of the approaches to this has been to consider the Bondi-Metzner-Sachs (BMS) group. In the absence of gravity, the isometry group of the spacetime is the well-known Poincaré group which is the semidirect product of translations and Lorentz transformations. The situation, however, changes drastically when gravity is turned on, even for weak gravitational fields. When one looks at four-dimensional asymptotically flat metrics, the isometry group of the background metric is enhanced to an infinite dimensional asymptotic symmetry group at null infinity. This is the BMS group [3]. This consists of the semidirect product of the global conformal group of the unit 2-sphere and the infinite dimensional "supertranslations". There is a further enhancement to two copies of the centerless Virasoro or the Witt algebra times the supertranslations if one does not require the transformations generated to be well defined [4]. In three dimensions $\mathfrak{bms}_3$ is again infinite dimensional and now has one copy of the Witt algebra along with the supertranslations [5]. These correspond to the null boundary of the three dimensional spacetime which is $S^1 \times \mathrm{R}$. Let us recover the BMS group from the symmetries of flat spacetime. To find the asymptotic symmetries we would need to look at the structure at null infinity. Let us begin by introducing the retarded time $u = t - r$, the luminosity distance r and angles θ^A on the $n-2$ sphere by $x^1 = r\cos\theta^1$, $x^A = r\sin\theta^1, \dots \sin\theta^{A-1}\cos\theta^A$, for $A = 2, \dots n-2$, and $x^{n-1} = r\sin\theta^1, \dots \sin\theta^{n-2}$. The Minkowski metric is then given by

$$d\bar{s}^2 = -du^2 - 2dudr + r^2 \sum_{A=1}^{n-2} s_A (d\theta^A)^2, \qquad (1)$$

where $s_1 = 1$, $s_A = \sin^2\theta^1, \dots \sin^2\theta^{A-1}$ for $2 \le A \le n-2$. The (future) null boundary is defined by $r = \text{constant} \to \infty$ with u, θ^A held fixed. One requires asymptotic Killing vectors to satisfy the Killing equation to leading order. They turn out to be [5]

$$\begin{aligned} \xi^u &= T(\theta^A) + u\partial_1 Y^1(\theta^A) + \mathcal{O}(r^0), \\ \xi^r &= -r\partial_1 Y^1(\theta^A) + \mathcal{O}(r), \\ \xi^A &= Y^A(\theta^B) + \mathcal{O}(r^0), \qquad (A = 1, \dots n-2.) \end{aligned} \qquad (2)$$

where $T(\theta^A)$ is an arbitrary function on the $n-2$ sphere, and $Y^A(\theta^A)$ are the components of the conformal Killing vectors on the $n-2$ sphere. These asymptotic Killing vectors form a subalgebra of the Lie algebra of vector fields and the bracket induced by the Lie bracket $\hat{\xi} = [\xi, \xi']$ is determined by

$$\begin{aligned} \hat{T} &= Y^A\partial_A T' + T\partial_1 Y'^1 - Y'^A\partial_A T - T'\partial_1 Y^1, \\ \hat{Y}^A &= Y^B\partial_B Y'^A - Y'^B\partial_B Y^A. \end{aligned} \qquad (3)$$

The asymptotic Killing vectors with $T = 0 = Y^A$ form an Abelian subalgebra in the algebra of asymptotic Killing vectors. The quotient algebra is defined to be $\mathfrak{bms}_n$. It is the semidirect sum of the conformal Killing vectors Y^A of Euclidean $n-2$ dimensional space with an Abelian ideal of so-called infinitesimal supertranslations. In three dimensions, the conformal Killing equation on the circle imposes no restrictions on the function $Y(\theta)$. Therefore, $\mathfrak{bms}_3$ is characterized by 2 arbitrary functions $T(\theta)$, $Y(\theta)$ on the circle. These functions can be Fourier analyzed by defining $P_n \equiv \xi[T = \exp(in\theta), Y = 0]$ and $J_n = \xi[T = 0, Y = \exp(in\theta)]$. In terms of these generators, the commutation relations of $\mathfrak{bms}_3$ become (we drop all factors of i)

$$\begin{aligned} &[J_m, J_n] = (m-n)J_{m+n}, \qquad [P_m, P_n] = 0, \\ &[J_m, P_n] = (m-n)P_{m+n}. \end{aligned} \qquad (4)$$

The Galilean conformal algebra (GCA) on the other hand, has been discussed in literature in connection with a nonrelativistic limit of the AdS/CFT conjecture [6]. It was obtained by a parametric contraction of the finite conformal algebra (see for example Refs. [6,7]) and was observed to have an infinite dimensional lift for all spacetime dimensions [6]. The 2 and 3 point correlation functions of the GCA were found [8]. However, in two spacetime dimensions, as is well known, the situation is special. The relativistic conformal algebra is infinite dimensional and consists of two copies of the Virasoro algebra. One expects this to be also related, now to the infinite dimensional GCA algebra. In two dimensions the nontrivial generators are the L_n and the M_n:

$$L_n = -(n+1)t^n x\partial_x - t^{n+1}\partial_t, \qquad M_n = t^{n+1}\partial_x, \quad (5)$$

which obey

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [M_m, M_n] = 0,$$
$$[L_m, M_n] = (m-n)M_{m+n}. \quad (6)$$

The generators in (5) arise precisely from a nonrelativistic contraction of the two copies of the Virasoro algebra of the relativistic theory. The nonrelativistic contraction consists of taking the scaling

$$t \to t, \qquad x \to \epsilon x, \quad (7)$$

with $\epsilon \to 0$. This is equivalent to taking the velocities $v \sim \epsilon$ to zero (in units where $c = 1$). Consider the vector fields which generate (two copies of) the Witt algebra in two dimensions:

$$\mathcal{L}_n = -z^{n+1}\partial_z, \qquad \bar{\mathcal{L}}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \quad (8)$$

In terms of space and time coordinates, $z = t + x$, $\bar{z} = t - x$. Hence $\partial_z = \frac{1}{2}(\partial_t + \partial_x)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_t - \partial_x)$. Expressing $\mathcal{L}_n$, $\bar{\mathcal{L}}_n$ in terms of t, x and taking the above scaling (7) ($\epsilon \to 0$) reveals:

$$\mathcal{L}_n + \bar{\mathcal{L}}_n = -t^{n+1}\partial_t - (n+1)t^n x\partial_x + \mathcal{O}(\epsilon^2) \to \mathcal{L}_n,$$
$$\mathcal{L}_n - \bar{\mathcal{L}}_n = -\frac{1}{\epsilon}t^{n+1}\partial_x + \mathcal{O}(\epsilon) \to -\frac{1}{\epsilon}M_n. \quad (9)$$

Thus the GCA in $2d$ arises as the nonrelativistic limit of the relativistic algebra.

The $\mathfrak{bms}_3$-$\mathfrak{gca}_2$ *correspondence.*—From (4) and (6), it is obvious that the algebras $\mathfrak{bms}_3$ and $\mathfrak{gca}_2$ are isomorphic with the trivial identifications $L_n \leftrightarrow J_n$, $M_n \leftrightarrow P_n$. So, what we have is a holographic correspondence between an asymptotic three-dimensional flat spacetime and a two-dimensional nonrelativistic conformal field theory. There have been some different forms of realizations of the bulk theory which has the GCA as its boundary algebra. Originally in Ref. [6], we proposed the dual gravity theory to be a Newton-Cartan–like $AdS_2 \times R^d$ by taking a similar nonrelativistic limit on the bulk anti–de Sitter space. So, for the case of AdS_3, which is the focal point of our attention now, the theory in the bulk was a $AdS_2 \times R$ Newton-Cartan. The L_n's turned out to be the asymptotic symmetries in the sense of Brown and Henneaux [9]. (The M_n's were trivial isometries of the base AdS_2 and hence did not have any Brown-Henneaux–like interpretation.) It has also been observed that the GCA emerges as the asymptotic symmetry algebra of cosmological topologically massive gravity in three dimensions when the coefficient of the gravitational Chern-Simons term is made very large [10]. This realization of the GCA in the bulk allows for an asymmetry which is required in the central charges of the quantum GCA. The new gravity description in terms of the BMS algebra gives a third and possibly the most intriguing occurrence of the GCA. Some might argue that in order to really have this correspondence, we would need a concrete realization of the boundary theory. To answer this question it is to be noted that the infinite $2d$ GCA also makes its appearance in nonequilibrium statistical mechanical systems [11]. So, there is a candidate for the CFT which satisfies the requirements of the symmetry and it is possible that this could be a realization of the BMS-GCA correspondence.

The Euler equation in nonrelativistic hydrodynamics emerges in situations when the viscosity of the fluid is negligible. In Ref. [6], it was noted that the finite dimensional GCA is the symmetry algebra of the Euler equations. In fact, it is interesting that all the M_n's (for any n) are also symmetries of the equations [6,12]. In the introduction, we had remarked that in four dimensions, if the BMS group is not extended to include all conformal transformations, then it consists of the semidirect product of the global conformal group in two dimensions and the supertranslations. The situation is similar in three dimensions. We can look at the "restricted" BMS algebra, with only the global part of the conformal transformations included. This contains $L_{\{0,\pm1\}}$ together with all the M_n's. So, yet another curious observation of the BMS-GCA correspondence is that the features of the restricted BMS group in 3 dimensions is encoded in the symmetries of the Euler equations in 1 + 1 dimensions.

Let us also briefly comment on the central charges on both sides of this correspondence. A more detailed discussion of this can be found in [13] which is an extended version of this Letter. The quantum mechanical 2D GCA allows two types of Virasoro-like central charges C_1, C_2 for the $[L, L]$ and $[L, M]$ commutators [14]. The gravity calculations in Ref. [5] and the BMS-GCA correspondence implies that in this context $C_1 = 0$ and $C_2 = 1/4G$. These can also be motivated by a limit from AdS_3 [13].

A BMN route to BMS.—In Ref. [6], we proposed a gravity dual of the GCA by taking a parametric limit of the bulk AdS_{d+2} geometry. Consider the metric of AdS_{d+2} in Poincaré coordinates

$$ds^2 = \frac{1}{z'^2}(dt'^2 - dz'^2 - dx_i^2). \quad (10)$$

The nonrelativistic scaling limit that was considered was

$$t', z' \to t', z' \qquad x_i \to \epsilon x_i. \quad (11)$$

The scalings of t and x_i were motivated by the boundary scaling. The radial direction of the AdS_{d+2} is an additional dimension.We fix its scaling by remembering that it is a measure of the energy scales in the boundary theory via the usual holographic correspondence and should scale like time, i.e., as ϵ^0. So, in the bulk the time and radial directions *both* survive the scaling and constitute an AdS_2 sitting inside the original AdS_{d+2}. The GCA was shown to emerge by taking this limit on the Killing vectors of AdS_{d+2}. Let us suggest a different bulk realization of the GCA relevant to the asymptotically flat spaces as opposed to the Newton-Cartan structure proposed in Ref. [6].

Let us concentrate on AdS_3. We will now reintroduce factors of the AdS radius R. We would take a Penrose limit of the AdS metric in the co-ordinates stated above. (This was obtained in collaboration with Rajesh Gopakumar.) The Poincaré patch has a horizon at $z' = \infty$ and to extend the coordinates beyond this we will choose to follow an infalling null geodesic, in an analogue of the Eddington-Finkelstein coordinates. Therefore define $z = z'$ and $t = t' + z'$. In these coordinates, with the radius included the bulk metric reads:

$$ds^2 = \frac{R^2}{z^2}[-dt(2dz - dt) - dx^2] \tag{12}$$

This will give a nondegenerate metric only if we have the scaling

$$x = \frac{\mu}{R}x, \qquad t, z \sim \mathcal{O}(1), \qquad t - 2z = \frac{\mu^2}{R^2}v \tag{13}$$

with $R \to \infty$ and keeping μ, v, x_i, t finite. The resulting metric is

$$ds^2 = \frac{4\mu^2}{t^2}(dtdv - dx^2) \tag{14}$$

where we have kept the leading order terms as $R \to \infty$. (Notice $z^2 = t^2 + \mathcal{O}(R^{-2})$ and hence the replacement.) This is like a BMN limit [15] where we are zooming into the vicinity of the null radial geodesic. Note that $t - 2z = t' - z'$ in terms of the original Poincaré coordinates that we started out with.

However, this metric is actually flat. This can be seen by writing $x = t\rho$. This gives the metric

$$\frac{ds^2}{4\mu^2} = \frac{dz}{t^2}(dv - \rho^2 dt - 2\rho t d\rho) - d\rho^2$$
$$= -d\left(\frac{1}{t}\right)d(v - \rho^2 t) - d\rho^2. \tag{15}$$

which we see is clearly a flat metric on $R^{2,1}$ when we define $\tilde{u} = \frac{1}{t}$ and $\tilde{v} = v - \rho^2 t$. We have kept only the leading terms in the above computation. The above computation shows that by taking a nonrelativistic limit (13) which is almost exactly like (11), with an additional condition on $t - z$, we can recover a flat space. It was shown in Ref. [6], that the Killing vectors of AdS in the limit (11) give rise to an infinite algebra in the bulk which precisely reduces to the GCA on the boundary and satisfy the same commutation relations in the bulk. Given the connection between $\mathfrak{bms}_3$ and $\mathfrak{gca}_2$, it is satisfying that one has been able to recover a flat space metric using the same limit. In terms of the AdS_3/CFT_2 correspondence, the above mentioned BMS_3-GCA_2 is thus a limit where on the gravity side one takes the radius of AdS to infinity while on the field theory side, one takes the speed of light to infinity. So, this seems to indicate an equivalence between the radius of AdS and the speed of light in the CFT.

The $\mathfrak{bms}_4$-$\mathfrak{gca}_3^{\mathfrak{s}=1}$ *correspondence.*—**The BMS group in 4 dimensions:** The structure of the BMS group in four dimensions as before is dictated by the structure at null infinity, which is now $S^2 \times R$ times the supertranslations. As in the case of the three-dimensional BMS group, if one does not want to restrict to globally well-defined transformations on the two-sphere, we get two copies of the Witt algebra. The general solution to the conformal Killing equations is $Y^\zeta = Y(\zeta)$, $Y^{\bar{\zeta}} = \bar{Y}(\bar{\zeta})$, with Y and $\bar{Y}$ independent functions of their arguments. Their standard basis vectors and the generators of the supertranslations:

$$l_n = -\zeta^{n+1}\frac{\partial}{\partial\zeta}, \qquad \bar{l}_n = -\bar{\zeta}^{n+1}\frac{\partial}{\partial\bar{\zeta}}, \qquad n \in \mathbb{Z}$$

$$T_{m,n} = P^{-1}\zeta^m\bar{\zeta}^n, m, n \in \mathbb{Z}. \qquad [P(\zeta,\bar{\zeta}) = \frac{1}{2}(1 + \zeta\bar{\zeta})].$$

In terms of the basis vector $l_l \equiv (l_l, 0)$ and $T_{mn} = (0, T_{mn})$, the commutation relations for the complexified $\mathfrak{bms}_4$ algebra read

$$[l_m, l_n] = (m-n)l_{m+n}, \qquad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}$$
$$[l_l, T_{m,n}] = \left(\frac{l+1}{2} - m\right)T_{m+l,n}$$
$$[\bar{l}_l, T_{m,n}] = \left(\frac{l+1}{2} - n\right)T_{m,n+l}. \tag{16}$$

Two copies of the Witt algebra indicate that we would need to look beyond usual GCAs in any dimensions as the field theory realizations of this symmetry.

Semi-Galilean conformal algebras: In order to find a field theoretic description of the above symmetry, in this section we study nonrelativistic limit of relativistic conformal algebra in $d + 1$ dimensions by making use of a general contraction.

$$t \to t, \qquad y_\alpha \to y_\alpha, \qquad x_i \to \epsilon x_i, \tag{17}$$

where $\alpha = 1, \ldots s$ and $i = s + 1, \ldots d$. The contraction is defined by the above scaling in the limit of $\epsilon \to 0$.

We start from a CFT in $d + 1$ dimensions. The GCA was a specific example of the semi-GCA with $s = 0$. We label semi-GCAs by $\mathfrak{gca}_{\mathfrak{n}}^{\mathfrak{s}}$. (In our notation, GCA is $\mathfrak{gca}^{\mathfrak{s}=0}$.) Let us consider the case of $s = 1$, $d = 2$. As in the $s = 0$ case, there is a finite algebra which is obtained by contraction and then this can be given an infinite dimensional lift. It is useful to define new coordinates $u = t + y$, $v = t - y$. The infinite generators are [16]

$$L_n = u^{n+1}\partial_u + \frac{n+1}{2}u^n x\partial_x, \qquad M_{rs} = -u^r v^s \partial_x.$$
$$\bar{L}_n = v^{n+1}\partial_v + \frac{n+1}{2}v^n x\partial_x. \tag{18}$$

One finds an infinite dimensional algebra as follows

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m},$$
$$[L_n, M_{ml}] = \{[(n+1)/2] - m\}M_{(n+m)l},$$
$$[\bar{L}_n, M_{ml}] = \{[(n+1)/2] - l\}M_{m(n+l)}. \tag{19}$$

It is straight forward to observe that this algebra which we call $\mathfrak{gca}_3^{\bar{s}=1}$ is isomorphic to (16). The dimension of the field theory is three and the coordinates are t, y, x, viz., one of each kind mentioned above.

Remarks on a general correspondence.—The BMS algebra is infinite dimensional in 3 and 4 dimensions as the group of isometries of the circle and the spheres are enhanced to the full conformal algebra. Even in the case of the GCA and its cousin, called the semi-GCA with $s = 1$ we have seen similar enhancements. For $s = 2$ and beyond, we do not get this enhancement. A way of understanding this fact is to remember the gravity description in terms of the Newton-Cartan structure, as defined in Ref. [6]. The GCA had a bulk piece with an AdS_2 and the $s = 1$ GCA had a similar AdS_3 counterpart. The higher sGCA would have AdS_{2+s} factors in the Newton-Cartan structure, but these would not have enhanced Virasoro-like symmetries. This suggests a general BMS_{3+s}-$s$$GCA_{2+s}$ correspondence. We should keep one nonrelativistic direction in the conformal field theory that is dual to the one higher dimensional BMS group. For example, when we are looking at the BMS algebra in five dimensions, we should look at a four-dimensional $s = 2$ GCA. As a limit of an AdS/CFT correspondence, one should easily be able to obtain the flat space limit of the respective AdS space by looking at higher dimensional analogues of the radially infalling null ray and scaling the coordinates in a manner described in the 3D case. Now all differences between relativistic coordinates would scale as R^{-2}. The speed of light in the nonrelativistic direction of the field theory would be the dual to the AdS radius in the bulk theory.

Conclusions.—In this note, we have looked at two seemingly unrelated pictures, that of asymptotically flat spaces and nonrelativistic conformal systems in one lower dimension and shown that they are equivalent at the level of the symmetry algebras. We have looked at the structure of the 2D nonrelativistic field theory in some detail in Ref. [14]. It was found there that most of the answers in the GCA can be obtained in a spirit very similar to the techniques of 2D conformal symmetry. Given this correspondence between flat space and the GCA, it is tempting to ponder on the consequences of our analysis in Ref. [14] and hope to make statements about answers on the gravity side. For this, a first step would be to identify the parameters on both sides. We had labeled the GCA with the eigenvalues of boosts and dilatations in Refs. [8,14]. It would be useful to understand the relations of these to physical quantities in the asymptotically flat spaces. The correlation functions of the GCA were also computed in Refs. [8,14]. The correlation functions in the field theory would map to on shell amplitudes in gravity. It is plausible that one would be able to make statements about the S matrix in asymptotically flat spacetimes by using the techniques of the GCA. We are unaware of any such analysis using the BMS algebra and this is an avenue definitely worth exploring.

The four-dimensional case, i.e. BMS_4-$s$$GCA_3$ is a case we have talked less about in this note. This is however the more interesting map for physical systems because of the obvious reason that we are talking about asymptotically flat four-dimensional space. On the field theory side, this is a case which has been far less studied. One must look at the representations and in a spirit similar to Ref. [8], one should be able to construct the Hilbert space and find the two and three point functions by looking at just the global part of the algebra. This note is a first step in the direction of using the nonrelativistic conformal techniques to study the holography of flat space, which we hope would be a worthwhile exercise.

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