Transformation Optics with Photonic Band Gap Media

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We introduce a class of optical media based on adiabatically modulated, dielectric-only, and potentially extremely low-loss, photonic crystals (PC). The media we describe represent a generalization of the eikonal limit of transformation optics (TO). The basis of the concept is the possibility to fit some equal frequency surfaces of certain PCs with elliptic surfaces, allowing them to mimic the dispersion relation of light in anisotropic effective media. PC cloaks and other TO devices operating at visible wavelengths can be constructed from optically transparent substances such as glasses, whose attenuation coefficient can be as small as 10 dB/km, suggesting the TO design methodology can be applied to the development of optical devices not limited by the losses inherent to metal-based, passive metamaterials.

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Transformation optics (TO) is a recently appreciated methodology for the design of novel electromagnetic (EM) devices [1,2]. Based on the invariance of Maxwell's equations with respect to arbitrary coordinate transformations, TO solves the inverse EM problem by suggesting the distribution of EM material properties that implement a desired field configuration. Devices created with the TO concept typically require exotic EM properties [3–5], which are unavailable in naturally occurring optical media such as solid-state crystals and amorphous substances (glasses). In fact, the required properties are so difficult to achieve, that TO-based designs would have been dismissed prior to the development of mesoscopic "metamaterial" fabrication methods [3,4,6]. The desired material properties are usually available in metamaterials only at certain frequencies in the vicinity of a resonance band, which makes them inherently lossy.

The lossy nature of metamaterials has impeded progress in certain areas of metamaterial and TO research, including subdiffraction limited imaging (superlensing) [7] and EM invisibility [1,5,6]. In the case of superlensing, which requires the exotic property of negative refractive index (RI), an alternative, low-loss solution was proposed based on the existence of negative group velocity bands in photonic band gap media, also known as photonic crystals (PC) [8–11]. PC-based imaging devices suffer from serious drawbacks, such as narrow operational bandwidth and inefficient coupling between free space and the PC [10–12]. However, they offer the possibility of extremely low optical loss. The optical attenuation in a photonic band gap medium may be so low that it could span millions of wavelengths and yet be virtually transparent; the glass used in optical fibers [13] is a famous example of such a low-loss dielectric component.

TO designs result in inherently complex media, typically requiring the exotic property of superluminal phase velocity (i.e., RI $n < 1$) combined with anisotropy (such as in beam shifters [12,14]), spatial gradients of RI (as in the optical black hole [15] and the flattened Luneburg lens [16]), or both anisotropy and gradients (as in the cloak of invisibility [1,6]); see Ref. [2] for a recent review. The "invisibility cloak," in particular, has stimulated considerable interest over the past several years, and serves as a natural test bed for various TO-inspired concepts and sophisticated metamaterial designs. In this article, we similarly design a material distribution that operates as an optical cloak in the ray-optics limit, to demonstrate the general ability to manipulate light propagation and mimic TO media with PCs. This all-dielectric device suitable for visible wavelengths is potentially closer to a real invisibility tool than the other dielectric-only "cloaking" structures [17,18], since the latter can create optical illusions only for a limited range of view angles.

In its full generality, TO considers a coordinate transformation (CT) of a volume with an arbitrary bounding surface onto another arbitrarily shaped volume [1,2]. For the invisibility cloak [1], one can restrict the CTs to radially symmetric maps. A perfect two-dimensional cloak can be obtained from a CT $r' = q(r)$ that maps the annulus $a < r < b$ onto the disk $0 < r' < b$ [1,6]. The constitutive parameters of the cloak in the physical space (with radial coordinate r) are prescribed as follows [19]:

$$\epsilon_r = \mu_r = \frac{q}{rq'}, \qquad \epsilon_\theta = \mu_\theta = 1/\epsilon_r,$$
$$\epsilon_z = \mu_z = \frac{qq'}{r}, \tag{1}$$

where $q' \equiv dq(r)/dr$. The RI corresponding to radial (n_r) and azimuthal (n_θ) propagation in the perfect cloak (1) equals $n_r = q'$ and $n_\theta = q/r$, respectively, for both TE [6] and TM [19] polarizations.

In the geometrical optics limit one may abandon the notion of ϵ and μ in favor of RI, since only the optical path length has relevance. The RI is defined as the wave number

k of a wave propagating in a certain direction, normalized to the vacuum wave number $k_0 = \omega/c$, i.e., $n(\vec{k}) = |\vec{k}|/k_0$. Knowing the function $n(\vec{k})$, or $\omega(\vec{k})$, is sufficient to describe the dynamics of light in transparent media [20,21]. Such a description is a general approach covering effective medium regimes, where constitutive tensors ϵ, μ can be introduced, as well as Bloch-Floquet (BF) regimes in periodic media, such as gratings [21] and PCs [8]. If the function $n(\vec{k}, \vec{r})$ changes slowly (adiabatically) with position $\vec{r}$, i.e., $|\nabla_{\vec{r}} n(\vec{k}, \vec{r})| \ll |\vec{k}|$, light propagation can be described with the semiclassical (WKB) approximation [21].

The adiabaticity of $n(\vec{k}, \vec{r})$ is violated at a sharp transition between a PC and free space, where RI alone is not sufficient to describe wave propagation. Whereas an effective medium description would enable the reflectance and transmittance to be calculated from an effective wave impedance $Z = \sqrt{\mu\epsilon^{-1}}$, the latter is generally an ill-defined concept for PCs. The lack of a well-defined wave impedance in PCs leads to the generation of multiple refracted and reflected beams (Bragg diffraction), and a dependence of reflectivity upon the microscopic details of the surface [10,22]. Recent studies of negatively refracting PCs with regards to their imaging properties show that improving the coupling between free-space and PC modes is possible [10,11,22], although very difficult due to lack of a general theoretical model for the coupling. Though not addressed here, the mode coupling problem is essentially unrelated with the dispersion engineering problem that we solve below.

Once the effective medium description is replaced by the $\omega(\vec{k}, \vec{r})$ description, two features potentially useful for optical TO media can be achieved. First, materials with ϵ or μ less than one, which are necessarily dispersive and lossy, are no longer needed. The dispersion relation of a PC near the Γ point ($\vec{k} = 0$) can be written as a Taylor expansion with even powers of k:

$$\omega^2(\vec{k}) = \omega_\Gamma^2 + v_r^2 k_r^2 + v_\theta^2 k_\theta^2 + O(k^4), \qquad (2)$$

or, equivalently, $\omega^2(\vec{k})/c_0^2 = \frac{k_r^2}{n_r^2} + \frac{k_\theta^2}{n_\theta^2} + O(k^4)$, where c_0 is the speed of light in vacuum and $n_{r,\theta}^2 = (1 - \omega_\Gamma^2/\omega^2)c_0^2/v_{r,\theta}^2$. In the above we assumed that the propagation band has a positive group velocity, and thus, $\omega > \omega_\Gamma$; similar formulas for a negative-refraction band can be obtained by changing the sign of $v_{r,\theta}^2$ in the expressions above. The semiaxes of the elliptical equal frequency surfaces (EFS) are given by $k_r^{\max} = n_r \omega/c_0$ and $k_\theta^{\max} = n_\theta \omega/c_0$. In the Γ-point ($\omega = \omega_\Gamma$), $n_\theta = n_r = 0$. The effective indexes $n_{\theta,r}$ are not related to any effective medium theory and, consequently, not restricted by Voigt-Reiss or Hashin-Shtrikman bounds arising from the classical homogenization theories [8]. Second, anisotropy of RI is possible in dielectric-only PCs, and not only for TM, but also TE waves. This contrasts with the behavior of TE waves in dielectric effective media, whose RI $n = \sqrt{\epsilon_z}$ is always isotropic.

While the approximately elliptic EFS (2) are very common in the band structure of PCs in the vicinity of band gaps [8,23,24], some propagation bands possess elliptic EFS in a relatively wide band. The existence of a broadband elliptic EFS in a one-dimensional PC formed by two alternating dielectric layers is illustrated by Fig. 1. The dispersion relation of the bilayer PC is known in analytical form [20]. Similar elliptic bands can be found in higher-dimensional PCs, whose dispersion relation can be computed numerically.

Theoretically, TO is a powerful framework that enables unprecedented control over the propagation of EM waves [1,2]. Anisotropy of the material properties implementing nontrivial CTs is a typical property of the TO-based designs. A relatively simple TO-based device that uses anisotropic media is a beam shifter [14]. Its simplicity is due to the fact that a transverse shift of the beam can be obtained with a piecewise linear CT, which results in piecewise constant material properties. Thus, the linear beam shifter can be implemented as a flat slab with a uniform, although anisotropic, effective index. The beam-shifting effect of BF modes was discovered in singly periodic planar waveguides without invoking TO [25]. Reflectionless beam shifters were later demonstrated theoretically in the effective medium regime and explained using the concept of embedded CT [14].

A linear beam shifter with the shift parameter $a = \tan(\phi)$ deflects the beam at an angle ϕ, resulting in a transverse shift $\Delta y = ad$, where d is the thickness of the shifter [14]. Below, we design and simulate a beam shifter operating in a higher propagation band of a PC. The TM-polarized BF mode of the bilayer PC illustrated and described in Fig. 1 is selected for this demonstration. The operational frequency is chosen such that the principal RIs of the beam shifter medium, $n_{1,2} = \{1 + a^2/2[1 \mp \sqrt{(1 + 4/a^2)}]\}^{1/2}$, match

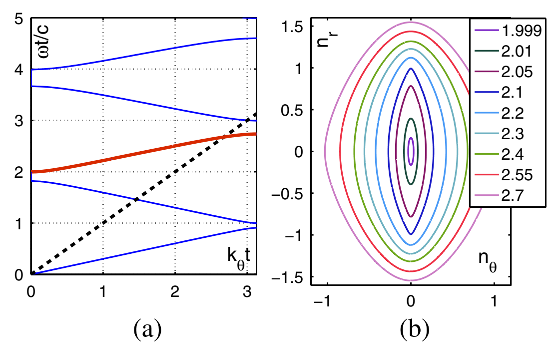

FIG. 1 (color online). Dispersion relation for TM waves in a bilayer PC with permittivities $\epsilon_1 = 12$, $\epsilon_2 = 1$ and filling fraction $t_1/t = 0.9$. (a) Band diagram, $\omega t/c \equiv k_0 t$ vs $k_\theta t$, plotted for zero phase shift in the nonperiodic direction ($k_r = 0$). Light line ($n_\theta = 1$) is indicated by dashes. (b) EFS of the highlighted band in the $k_\theta - k_r$ space, for several choices of frequency within the region with $0 \le n_\theta \le 1$. The axes are labeled by effective RI $n_\theta = k_\theta/k_0$ and $n_r = k_r/k_0$. Partially overlapping EFS of the adjacent band are not shown.

the RIs of the bilayer PC computed analytically [20]; this leads to the choice $k_0 t = 2.449$, where t is the PC period. The resulting beam shifter has shift parameter $a = 0.6187$ and shift angle $\phi = 31.7°$; RI principal values are $n_1 = 0.7374$ and $n_2 = 1.3561$. The principal direction of n_1 (the layer direction) makes an angle $\theta_1 = 36°$ with the horizontal axis.

The operation of the beam shifter is demonstrated using finite element frequency domain (FEFD) simulations performed with COMSOL [26]. For index-only TO devices, Gaussian beam is a natural probe of the design [15,25], as its path should agree with the ray trajectory in the geometrical optics limit. The propagation of a Gaussian beam incident horizontally on a slab of homogeneous anisotropic medium with the RIs described above is shown in Fig. 2(a). The beam propagates in the direction of its group velocity $\vec{v}_g$ in the PC, making the angle $\phi = 31.7° \neq \theta_1$ with the horizontal axis. The phase fronts remain vertical inside the shifter, indicating that the phase velocity stays horizontal in the PC. Figure 2(b) shows the propagation of a beam with the same parameters incident onto a PC slab with the same RI pair. To reduce internal reflections, the PC slab is terminated with absorbing layers matched perfectly to the PC, which are seen as the dark rectangles at the top or bottom of the PC domain in Fig. 2(b). Their function is to absorb the second (diffracted) beam excited on PC surface by a free-space Gaussian mode; the second beam corresponds to an intersection of the $k_y = 0$ line with the portions of EFS outside the range of wave numbers shown in Fig. 1(b).

To achieve a more complex TO design, such as a cloak, we may add adiabatic spatial modulation to the bilayer PC. The cylindrical cloak derived from Eqs. (1) requires $n_\theta < n_r$. We have found numerically that in all elliptic bands of the bilayer PC (except the lowest, *acoustic* branch), the RI for the propagation along the layers ($n_\|$) is always greater than in the stacking direction ($n_\perp$); this was found for both TM and TE waves. Note that this property of higher BF modes is opposite to the behavior

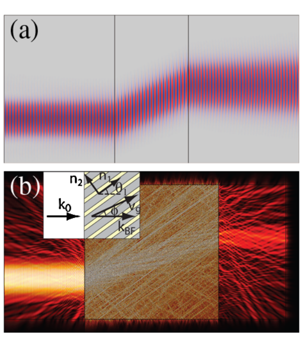

FIG. 2 (color online). Parallel-slab beam shifter based on an embedded CT. (a) Gaussian beam incident from vacuum on a slab of anisotropic effective medium. Color shows out-of-plane magnetic field $\mathrm{Re}(H_z)$. (b) Same as (a) but with a PC slab. Color shows $|H_z|$.

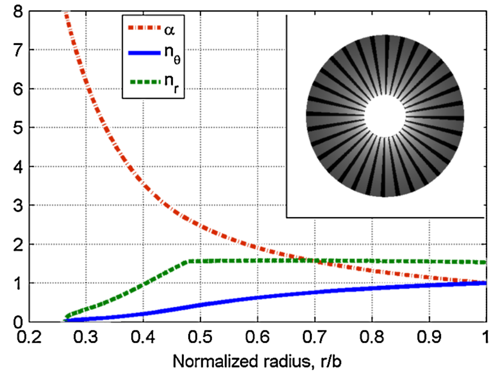

FIG. 3 (color online). TM-wave cloak based on the bilayer PC with $\epsilon_1/\epsilon_2 = 12$ and filling fraction $t_1/t = 0.9$. Dash-dotted line: dielectric constant scaling factor (α) vs normalized radius (r/b). Solid and dashed: effective indexes n_θ and n_r, respectively. Inset: the cloak geometry.

of the acoustic TM band, which has $n_\perp = \sqrt{\epsilon_\|} \geq n_\| = \sqrt{\epsilon_\perp}$, as dictated by the Voigt-Reiss bounds of homogenization theory [8]. This finding mandates that the PC layers must be stacked azimuthally for the cloak application, as shown in the inset to Fig. 3.

The procedure for generating CTs for index-only cloaks has been introduced recently in Ref. [27]. In this case, it must be amended to account for the additional geometric constraint. In contrast with effective medium-based cloaks, the unit cells in PC cloaks must change continuously between layers. Thus, in a rotationally invariant design, the PC period in the azimuthal direction t must scale linearly with the radius to preserve the number of unit cells per circumference: $t(r) = t_b \frac{r}{b}$. To design a cloak, we assume that, besides $t(r)$, there is one more parameter that affects the band structure. A simple yet nonunique choice of such a parameter is the scaling factor for both permittivities α such that $\epsilon_{1,2}(r) = \epsilon_{1,2}(b)\alpha(r)$, where $\alpha(b) = 1$. To find the function $\alpha(r)$ that gives a cloak, one may start from the differential equation of Ref. [27], $\frac{dr}{r} = \frac{dn_\theta}{n_r(\alpha,t) - n_\theta(\alpha,t)}$, where $n_{r,\theta}$ are known functions of α and t. Noticing that the band structure is a function of a single parameter, $\beta(r) = t(r)\alpha^{1/2}(r)$, the solution to the above ODE is found as a simple quadrature: $\int \frac{dr}{r} = \int \frac{d\beta}{n_r(\beta) - n_\theta(\beta)} \frac{dn_\theta}{d\beta}$. The resulting permittivity scaling and RIs for the bilayer PC with arbitrarily chosen dielectric contrast $\epsilon_1/\epsilon_2 = 12$ and filling ratio $t_1/t = 0.9$ are plotted in Fig. 3. The cloak with these parameters has the aspect ratio $a/b = 0.2631$.

Thus designed PC structure is simulated with the FEFD solver [26]. The model geometry consists of two concentric disks of radii $b = 0.4$ and $a = 0.105\,24 = 0.2631b$. The inner disk represents the cloaked object, on which the perfect electric conductor boundary condition is imposed. The space between the circles is filled with a bilayer PC with $N = 512$ radially converging unit cells. The period in θ direction at $r = b$ equals $t_b = 4.9087 \times 10^{-3}$, and the

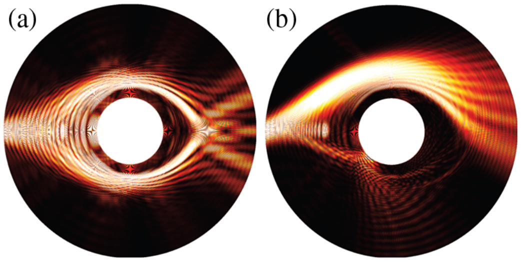

FIG. 4 (color online). (a) Cloaking of a normally incident Gaussian beam modulated with the desired BF mode. (b) Same as (a), but with the angle of incidence $\theta_{\mathrm{inc}} = 15°$.

normalized frequency of operation is $\omega t_b/c = 2.67\,755$. To suppress excitation of additional beams, a single BF mode is injected into the PC, by prescribing its local field distribution (spatially modulated with a Gaussian envelope) on the exterior boundary. The excited wave form is highly localized in the k space; it has no phase shift in the vertical direction and thus corresponds to a horizontally incident beam. As seen in Fig. 4(a), a Gaussian-like beam is created in the PC. The beam begins to diverge as it travels towards the center; then, it bends around the cloaked object and reforms as another beam on the opposite side of the PC structure. Excitation of the same mode with a nonzero phase shift per unit cell results in a Gaussian beam traveling at an angle, as shown in Fig. 4(b).

In conclusion, we have demonstrated theoretically and in the device-scale EM simulations that the concepts of transformation optics can be implemented with Bloch-Floquet waves in adiabatically modulated anisotropic photonic crystals, which can be composed of materials that are macroscopically transparent in the entire visible spectrum. This includes a demonstration of a dielectric-only, rotationally invariant structure that functions as an optical cloak in the ray-optics limit. This paradigm creates hope for a new class of optical-wavelength devices capable of being scaled to arbitrarily large dimensions.

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