Dissipative Relativistic Fluid Dynamics: A New Way to Derive the Equations of Motion from Kinetic Theory

G. S. Denicol,[1] T. Koide,[2] and D. H. Rischke[1,2]

[1]*Institute für Theoretische Physik, Johann Wolfgang Goethe-Universität, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany*
[2]*Frankfurt Institute for Advanced Studies, Johann Wolfgang Goethe-Universität, Ruth-Moufang-Str. 1, D-60438 Frankfurt am Main, Germany*

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We rederive the equations of motion of dissipative relativistic fluid dynamics from kinetic theory. In contrast with the derivation of Israel and Stewart, which considered the second moment of the Boltzmann equation to obtain equations of motion for the dissipative currents, we directly use the latter's definition. Although the equations of motion obtained via the two approaches are formally identical, the coefficients are different. We show that, for the one-dimensional scaling expansion, our method is in better agreement with the solution obtained from the Boltzmann equation.

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Dissipative relativistic fluid dynamics is an effective theory to describe the long-wavelength, low-frequency dynamics of various systems, with important applications in relativistic heavy-ion collisions and astrophysics [1]. However, the derivation of dissipative relativistic fluid dynamics from the underlying microscopic theory is not yet completely established.

It can be rigorously shown that, in the nonrelativistic, classical, dilute-gas limit, the Boltzmann equation becomes equivalent to the microscopic Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy equations [2]. In the relativistic case and/or for quantum fluids, a rigorous proof of this equivalence does not exist. Nevertheless, under certain approximations one can derive the Boltzmann equation from the Kadanoff-Baym equations [3]. Therefore, it is reasonable to assume that the Boltzmann equation constitutes a reliable approximation to the underlying microscopic dynamics, in particular, in the dilute-gas limit. Then, fluid dynamics can be systematically derived by introducing an appropriate coarse-graining scheme.

The Chapman-Enskog expansion [4] is the most common method to extract the fluid-dynamical equations of motion from the Boltzmann equation. However, this method is not suitable for relativistic systems, since it will inevitably lead to relativistic Navier-Stokes theory which displays intrinsic problems such as acausality and instabilities [5–7].

Israel and Stewart (IS) derived relativistic fluid-dynamical equations that do not exhibit this problem, by extending the method proposed by Grad for nonrelativistic systems [8]. In Grad's original work, the single-particle distribution function is expanded around its local equilibrium value in terms of a complete set of Hermite polynomials [9]. However, the generalization of Grad's approach to relativistic systems is nontrivial, since it is not easy to find a suitable set of orthogonal polynomials which could replace the Hermite polynomials [10,11]. Thus, Israel and Stewart introduced another approximation, the so-called 14-moment approximation [12], where the distribution function is expanded as a Taylor series in momentum around its local equilibrium value. The expansion is truncated at second order in momentum and only 14 coefficients remain to describe the distribution function.

It is important to note that the derivation of Israel and Stewart contains one additional approximation besides the 14 moments ansatz: they used the second moment of the Boltzmann equation to extract the equations of motion for the dissipative currents and, hence, to determine the transport coefficients [10–12]. However, this choice to extract the equations of motion is ambiguous, because *any* moment of the Boltzmann equation will lead to a closed set of equations, once the 14-moment approximation is applied. The transport coefficients appearing in the final equations depend on the choice of the moment.

Thus, the choice of the moment is quite an important issue [13]. In fact, it was confirmed that, at least for some cases, the IS equations are not in good agreement with the numerical solution of the Boltzmann equation [14,15]. Also, the transport coefficients obtained by Israel and Stewart do not coincide with quantum-field theoretical calculations [16]. These inconsistencies may arise because of an inappropriate choice of the moment equation.

Then, which moment should be used to derive fluid dynamics? Remember that we are interested in the equations of motion for the dissipative currents. Furthermore, the dissipative currents are well defined in terms of the single-particle distribution function from the kinetic point of view. Therefore, we can calculate the equations of motion for all the dissipative currents directly from their definitions without referring to an arbitrary moment of the Boltzmann equation. The purpose of this Letter is to derive new fluid-dynamical equations following this idea. We shall show that the form of these equations is the same as

in the IS theory, but the values of the coefficients are different. For the one-dimensional scaling expansion, we demonstrate that the new equations agree better with a numerical solution of the Boltzmann equation than the IS equations.

We start from the relativistic Boltzmann equation

$$K^\mu \partial_\mu f_K = C[f], \tag{1}$$

where $K^\mu = (E_{\mathbf{k}}, \mathbf{k})$ with $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ with m being the particle mass. In the collision term we consider only elastic two-to-two collisions,

$$C[f] = \frac{1}{2}\int dK' dP dP' W_{KK'\to PP'} \times (f_P f_{P'} \tilde{f}_K \tilde{f}_{K'} - f_K f_{K'} \tilde{f}_P \tilde{f}_{P'}). \tag{2}$$

Here, $dK \equiv g d^3\vec{K}/[(2\pi)^3 E_{\mathbf{k}}]$ is the Lorentz-invariant measure, with g being the degeneracy factor, and $W_{KK'\to PP'}$ is the transition rate of the collision. We used the notation $f_K \equiv f(x^\mu, K^\mu)$ and $\tilde{f}_K \equiv 1 - af(x^\mu, K^\mu)$, where $a = 1$ ($a = -1$) for fermions (bosons) and $a = 0$ for a Boltzmann gas.

The conserved particle current N^μ and the energy-momentum tensor $T^{\mu\nu}$ are expressed in terms of the single-particle distribution function as

$$N^\mu = \langle K^\mu \rangle, \tag{3}$$

$$T^{\mu\nu} = \langle K^\mu K^\nu \rangle, \tag{4}$$

where $\langle \cdots \rangle \equiv \int dK(\cdots) f_K$.

We introduce the fluid four-velocity u^μ as an eigenvector of the energy-momentum tensor, $T^{\mu\nu}u_\mu = \varepsilon u^\nu$, where the eigenvalue ε is the energy density [17]. Then, we can decompose the four-momentum as

$$K^\mu = (u\cdot K)u^\mu + K^{\langle\mu\rangle}. \tag{5}$$

Here, we defined the scalar product of two four-vectors A^μ, B^μ as $A_\mu B^\mu \equiv A\cdot B$ and we introduced the projection operator $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and $A^{\langle\mu\rangle} = \Delta^{\mu\nu}A_\nu$ for an arbitrary four-vector A^μ. The metric tensor is $g^{\mu\nu} \equiv \mathrm{diag}(+,-,-,-)$.

Using this decomposition, N^μ and $T^{\mu\nu}$ can be written in the form,

$$\begin{aligned} N^\mu &= nu^\mu + n^\mu, \\ T^{\mu\nu} &= \varepsilon u^\mu u^\nu - \Delta^{\mu\nu}(P + \Pi) + \pi^{\mu\nu}, \end{aligned} \tag{6}$$

where the particle density n, the particle diffusion current n^μ, the energy density ε, the shear stress tensor $\pi^{\mu\nu}$, and the sum of thermodynamic pressure, P_0, and bulk viscous pressure, Π, are defined by

$$\begin{aligned} &n \equiv \langle u\cdot K\rangle, \qquad n^\mu \equiv \langle K^{\langle\mu\rangle}\rangle, \qquad \varepsilon \equiv \langle (u\cdot K)^2\rangle, \\ &\pi^{\mu\nu} \equiv \langle K^{\langle\mu}K^{\nu\rangle}\rangle, \qquad P_0 + \Pi \equiv -\frac{1}{3}\langle \Delta^{\mu\nu}K_\mu K_\nu\rangle, \end{aligned} \tag{7}$$

where $A^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu\alpha\beta}A_{\alpha\beta}$, with $\Delta^{\mu\nu\alpha\beta} \equiv (\Delta^{\mu\alpha}\Delta^{\beta\nu} + \Delta^{\nu\alpha}\Delta^{\beta\mu} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta})/2$. We define the local equilibrium distribution function as $f_{0K} = [\exp(\beta_0 u\cdot K - \alpha_0) + a]^{-1}$, where β_0 and α_0 are the inverse temperature and the ratio of the chemical potential to temperature, respectively. These are defined by the matching conditions

$$n \equiv n_0 = \langle u\cdot K\rangle_0, \qquad \varepsilon \equiv \varepsilon_0 = \langle (u\cdot K)^2\rangle_0, \tag{8}$$

where $\langle\cdots\rangle_0 \equiv \int dK(\cdots)f_{0K}$.

The separation between thermodynamic pressure and bulk viscous pressure is then achieved by

$$P_0 = -\frac{1}{3}\langle \Delta^{\mu\nu}K_\mu K_\nu\rangle_0, \qquad \Pi = -\frac{1}{3}\langle \Delta^{\mu\nu}K_\mu K_\nu\rangle_\delta, \tag{9}$$

with $\langle\cdots\rangle_\delta \equiv \langle\cdots\rangle - \langle\cdots\rangle_0$.

So far, there is no difference between the calculation of Israel and Stewart and ours. The difference emerges in the derivation of the equations of motion for the dissipative currents. Israel and Stewart obtained these equations from the second moment of the Boltzmann equation [11,12]

$$\partial_\mu \langle K^\mu K^\nu K^\lambda\rangle = \int dK K^\nu K^\lambda C[f]. \tag{10}$$

Then, the equations of motion for Π, n^μ, and $\pi^{\mu\nu}$ are obtained by the projections $u_\nu u_\lambda \partial_\mu \langle K^\mu K^\nu K^\lambda\rangle$, $\Delta^\alpha_\lambda u_\nu \partial_\mu \langle K^\mu K^\nu K^\lambda\rangle$, and $\Delta^{\alpha\beta}_{\nu\lambda}\partial_\mu\langle K^\mu K^\nu K^\lambda\rangle$, respectively, together with the 14-moment approximation for the single-particle distribution function (see below). These equations determine the time evolution of Π, n^μ, and $\pi^{\mu\nu}$ through their comoving derivatives, $\dot{\Pi}$, $\dot{q}^{\langle\mu\rangle} \equiv \Delta^{\mu\nu}\dot{q}_\nu$, and $\dot{\pi}^{\langle\mu\nu\rangle} \equiv \Delta^{\mu\nu\alpha\beta}\dot{\pi}_{\alpha\beta}$, respectively, where $\dot{A} \equiv u\cdot\partial A$ is the comoving derivative.

However, we can calculate these comoving derivatives also directly from Eq. (7):

$$\dot{\Pi} = -\frac{1}{3}m^2\int dK \delta\dot{f}, \tag{11}$$

$$\dot{n}^{\langle\mu\rangle} = \int dK K^{\langle\mu\rangle}\delta\dot{f}, \tag{12}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \int dK K^{\langle\mu}K^{\nu\rangle}\delta\dot{f}. \tag{13}$$

Then, using the Boltzmann Eq. (1) in the form

$$\delta\dot{f} = -\dot{f}_0 - \frac{1}{u\cdot K}K\cdot\nabla f + \frac{1}{u\cdot K}C[f], \tag{14}$$

where $\nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$, we obtain the *exact* equations

$$\begin{aligned} \dot{\Pi} = &-C - \beta_\Pi\theta - \zeta_{\Pi\Pi}\Pi\theta + \zeta_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} \\ &- \zeta_{\Pi n}\partial\cdot n + \frac{m^2}{3}\nabla_\nu\langle (u\cdot K)^{-1}K^{\langle\nu\rangle}\rangle_\delta \\ &+ \frac{m^2}{3}\langle (u\cdot K)^{-2}K^\mu K^\nu\rangle_\delta \nabla_\mu u_\nu, \end{aligned} \tag{15}$$

$$\begin{aligned} \dot{n}^{\langle\mu\rangle} = &C^\mu + \beta_n\nabla^\mu\alpha_0 - n^\mu\theta - n\cdot\nabla u^\mu + h_0(\Pi\dot{u}^\mu - \nabla^\mu\Pi) \\ &+ h_0\Delta^{\mu\nu}\partial_\lambda\pi^\lambda_\nu - \Delta^\mu_\nu\nabla_\alpha\langle (u\cdot K)^{-1}K^{\langle\nu\rangle}K^{\langle\alpha\rangle}\rangle_\delta \\ &- \langle (u\cdot K)^{-2}K^{\langle\mu\rangle}K^\alpha K^\beta\rangle_\delta\nabla_\alpha u_\beta, \end{aligned} \tag{16}$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = C^{\mu\nu} + 2\beta_\pi\sigma^{\mu\nu} - \frac{5}{3}\pi^{\mu\nu}\theta - 2\pi_\rho^{\langle\mu}\sigma^{\nu\rangle\rho} + 2\pi_\rho^{\langle\mu}\omega^{\nu\rangle\rho} + 2\Pi\sigma^{\mu\nu} - \Delta^{\mu\nu}_{\lambda\sigma}\nabla_\rho\langle(u\cdot K)^{-1}K^{\langle\lambda\rangle}K^{\langle\sigma\rangle}K^{\langle\rho\rangle}\rangle_\delta - \langle(u\cdot K)^{-2}K^{\langle\mu}K^{\nu\rangle}K^\alpha K^\beta\rangle_\delta\nabla_\alpha u_\beta, \quad (17)$$

where $h_0 = n_0/(\varepsilon_0 + P_0)$, and we introduced the vorticity $\omega_{\lambda\rho} \equiv \frac{1}{2}(\nabla_\lambda u_\rho - \nabla_\rho u_\lambda)$, the shear tensor $\sigma_{\lambda\rho} \equiv \nabla_{\langle\lambda}u_{\rho\rangle}$ and the expansion scalar $\theta \equiv \nabla_\mu u^\mu$. Above, we used the following notation for the collision terms,

$$C = \frac{m^2}{3}\int dK(u\cdot K)^{-1}C[f],$$
$$C^\mu = \int dK(u\cdot K)^{-1}K^{\langle\mu\rangle}C[f], \quad (18)$$
$$C^{\mu\nu} = \int dK(u\cdot K)^{-1}K^{\langle\mu}K^{\nu\rangle}C[f].$$

However, because the remaining terms in angular brackets cannot be entirely expressed in terms of the macroscopic variables (7), Eqs. (15)–(17), are not closed. In order to obtain a closed set of equations, we use the 14-moment approximation for the single-particle distribution function introduced by Israel and Stewart

$$f_K = f_{0K} + f_{0K}\tilde{f}_{0K}(\lambda_\Pi\Pi + \lambda_n n_\alpha K^\alpha + \lambda_\pi\pi_{\alpha\beta}K^\alpha K^\beta), \quad (19)$$

and insert this into Eqs. (15)–(17), to compute the terms in angular brackets. This system of equations is now closed, since the approximation (19) solely involves the quantities of Eq. (7). The coefficients λ_Π, λ_n and λ_π are well-known functions of $u\cdot K$, α_0, and β_0, see e.g., Refs. [10–12] for details.

We finally obtain the equations of dissipative relativistic fluid dynamics,

$$\dot{\Pi} = -\frac{\Pi}{\tau_\Pi} - \beta_\Pi\theta - \ell_{\Pi n}\partial\cdot n - \tau_{\Pi n}n\cdot\dot{u} - \delta_{\Pi\Pi}\Pi\theta - \lambda_{\Pi n}n\cdot\nabla\alpha_0 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \quad (20)$$

$$\dot{n}^{\langle\mu\rangle} = -\frac{n^\mu}{\tau_n} + \beta_n\nabla^\mu\alpha_0 - n_\nu\omega^{\nu\mu} - \delta_{nn}n^\mu\theta - \ell_{n\Pi}\nabla^\mu\Pi + \ell_{n\pi}\Delta^{\mu\nu}\partial_\lambda\pi^\lambda_\nu + \tau_{n\Pi}\Pi\dot{u}^\mu - \tau_{n\pi}\pi^\mu_\nu\dot{u}^\nu - \lambda_{nn}n^\nu\sigma^\mu_\nu + \lambda_{n\Pi}\Pi\nabla^\mu\alpha_0 - \lambda_{n\pi}\pi^{\mu\nu}\nabla_\nu\alpha_0, \quad (21)$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} - \tau_{\pi n}n^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi n}\nabla^{\langle\mu}n^{\nu\rangle} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \lambda_{\pi n}n^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}. \quad (22)$$

The derived equations contain 25 transport coefficients, of which we only show the following three coefficients explicitly,

$$\beta_\Pi = \left(\frac{1}{3} - c_s^2\right)(\varepsilon_0 + P_0) - \frac{2}{9}(\varepsilon_0 - 3P_0) - \frac{m^4}{9}\langle(u\cdot K)^{-2}\rangle_0, \quad (23)$$

$$\beta_n = \frac{2}{3\beta_0}\langle 1\rangle_0 + \frac{m^2}{3\beta_0}\langle(u\cdot K)^{-2}\rangle_0 - \frac{n_0}{\beta_0}h_0, \quad (24)$$

$$\beta_\pi = \frac{4}{5}P_0 + \frac{1}{15}(\varepsilon_0 - 3P_0) - \frac{m^4}{15}\langle(u\cdot K)^{-2}\rangle_0, \quad (25)$$

where the velocity of sound (squared) is $c_s^2 = (dP_0/d\varepsilon_0)_{s_0/n_0}$ where s_0 is the entropy density. The other coefficients will be reported in Ref. [18]. While the form of the derived Eqs. (20)–(22), are the same as those obtained in previous calculations [19], the transport coefficients are different. That is, the derivation of the equations of dissipative relativistic fluid dynamics from the Boltzmann equation is ambiguous and depends on the method applied. However, in the nonrelativistic (low-temperature) limit, the set of transport coefficients as computed with the method of Israel and Stewart and ours converge to the same values. We remark that the equations of Ref. [19] were derived from the second moment of the Boltzmann equation just as the original IS equations [12]. Nevertheless, they contain additional terms that do not appear in Ref. [12]. The reason is that the power-counting scheme of Ref. [19] differs from the one employed by IS [12]. Despite this difference, we still refer to the equations of Ref. [19] as IS equations in the following.

Now we would like to quantify the difference between the IS equations and ours at hand of a simple example. We consider a massless Boltzmann gas equation of state and the one-dimensional Bjorken scaling expansion, where the velocity is given by $u^\mu = \frac{1}{\tau}(t, 0, 0, z)$, and the fluid-dynamical variables are only a function of the proper time τ. Then, Π and n^μ vanish. The shear stress tensor $\pi^{\mu\nu}$ has only diagonal components and can be characterized by a function π as $\pi^{\mu\nu} = \mathrm{diag}(0, \pi/2, \pi/2, -\pi)$.

The equation for π is given by

$$\frac{d\pi}{d\tau} + \frac{\pi}{\tau_\pi} = \beta_\pi\frac{4}{3\tau} - \lambda\frac{\pi}{\tau}. \quad (26)$$

In the massless limit, our transport coefficients simplify,

$$\beta_\pi = \frac{4P_0}{5}, \qquad \tau_\pi^{-1} = \frac{3}{5}\sigma P_0\beta_0, \qquad \lambda \equiv \frac{4}{9}\tau_{\pi\pi} + \delta_{\pi\pi} = \frac{124}{63}, \quad (27)$$

where σ is the total cross section [20]. Here we assumed that σ is independent of energy and momentum as is done in Refs. [14,15]. As mentioned above, the form of Eq. (27) is identical to that of the IS theory, but the transport coefficients assume different values. In the IS theory, these coefficients are given by

$$\beta_\pi = \frac{2P_0}{3}, \qquad \tau_\pi^{-1} = \frac{5}{9}\sigma P_0\beta_0, \qquad \lambda = 2. \quad (28)$$

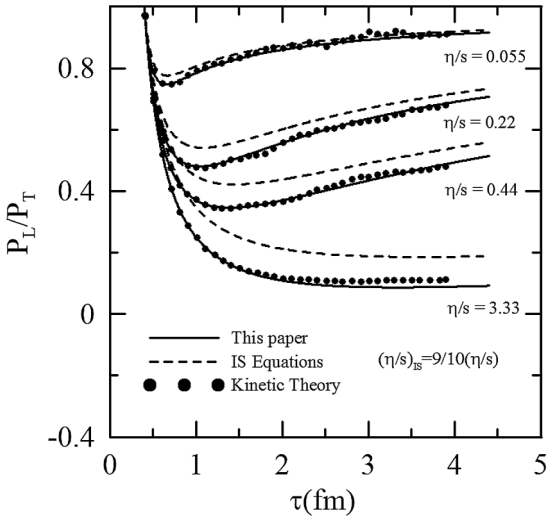

FIG. 1. Time evolution of the ratio P_L/P_T for our equations (solid line), the IS equations (dashed line), and the numerical solution of the Boltzmann equation (circles).

Equation (26) couples to the equation of the pressure which is given by

$$\frac{dP_0}{d\tau} + \frac{4P_0}{3\tau} - \frac{\pi}{3\tau} = 0. \tag{29}$$

In Fig. 1, we show the time dependence of the anisotropy of the effective pressure, which is defined by

$$\frac{P_L(\tau)}{P_T(\tau)} = \frac{P_0(\tau) - \pi(\tau)}{P_0(\tau) + \pi(\tau)/2}. \tag{30}$$

We used $T = 500$ MeV and $\pi = 0$ as initial condition. The solid and dashed lines represent our result and the result of the IS theory, respectively. The circles correspond to the numerical solution of the Boltzmann equation [15]. This calculation is performed with values for the cross section such that the shear viscosity $\eta \equiv \beta_\pi \tau_\pi$ to entropy density s ratio is constant. Since all the results must be compared by fixing a common cross section, $\eta = 4/(3\sigma\beta_0)$ and $\eta_{\rm IS} = 6/(5\sigma\beta_0)$ (the shear viscosity of the IS theory) have different values and are related by $\eta = \frac{10}{9}\eta_{\rm IS}$.

One can see that the IS theory always overestimates the anisotropy obtained by the numerical solution of the Boltzmann equation, even for very low viscosities ($\eta_{\rm IS}/s = 0.05$). On the other hand, our equations clearly show a better agreement. Visible deviations are only observed for the case of $\eta/s = 3.33$ at late times. This result indicates that our fluid-dynamical approach is better adapted than the IS theory to capture the microphysics contained in the Boltzmann equation.

In summary, we have proposed a new method for deriving the fluid-dynamical equations from kinetic theory. In our approach, the equations for the dissipative currents are obtained directly from the definitions of these currents. This method is different from the traditional IS approach [12], where the equations are extracted from the second moment of the Boltzmann equation. Our method can successfully reproduce the numerical solution of the Boltzmann equation for the simple one-dimensional scaling expansion. It is also important to mention that the transport coefficients of our kinetic calculation are consistent with those calculated from quantum field theory with the method proposed in Ref. [16].

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