Yangian-Invariant Scattering Amplitudes in Supersymmetric Chern-Simons Theory

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(Received 3 August 2010; published 5 October 2010)

We propose a generating function for scattering amplitudes of $\mathcal{N} = 6$ supersymmetric-Chern-Simons theory, which parallels a recent work on $\mathcal{N} = 4$ supersymmetric-Yang-Mills theory by Arkani-Hamed *et al.* Our result suggests that the scattering amplitudes of the supersymmetric-Chern-Simons theory exhibit Yangian invariance.

DOI: 10.1103/PhysRevLett.105.151603 PACS numbers: 11.15.Yc, 02.30.Ik, 11.30.Pb

Introduction.—The last decade has seen remarkable advances in novel methods for computing scattering amplitudes in perturbative Yang-Mills theories; see, e.g., [1,2] for recent reviews. While the new techniques are applicable for many theories including QCD, the $\mathcal{N} = 4$ supersymmetric-Yang-Mills theory (SYM_4) has proved to be the richest testing ground for new theoretical ideas.

Recently, Arkani-Hamed *et al.* [3] proposed a remarkably simple reformulation of the scattering amplitudes of planar SYM_4. They presented a "generating function for scattering amplitudes" named $\mathcal{L}_{n,k}$ in the form of a matrix-valued contour integral. With a suitable choice of the integration contour, $\mathcal{L}_{n,k}$ was conjectured to capture the leading singularities associated to n-point, k negative helicity amplitudes. In particular, it has been proven that $\mathcal{L}_{n,k}$ reproduces all tree-level amplitudes $\mathcal{A}^{\text{tree}}_{n,k}$ [4]. The applicability of the formulation of [3] to loop amplitudes is less clear; see [5] for the most recent progress.

One of the most striking features of SYM_4, which looks mysterious from traditional points of view but becomes transparent in the new formulation, is the so-called dual superconformal symmetry [6–9]. The original and dual superconformal symmetry of SYM_4 together generate a Yangian symmetry [10], which introduces elements of integrability into the scattering amplitudes of SYM_4 in the planar sector [11].

It is clearly interesting to see if the new formulation of [3] with a built-in Yangian symmetry can be applied to other field theories. One strong candidate is the three-dimensional $\mathcal{N} = 6$ super-Chern-Simons theory (SCS_6) constructed in [12–14]. This theory is special because of two common features it shares with SYM_4; it has a string theory dual in the sense of [15] and its superconformal algebra admits a simple extension to Yangian algebra as explained in [10].

Preliminary studies on scattering amplitudes of SCS_6 [16–18] reported results in favor of Yangian symmetry. These findings came as a surprise since earlier attempts had come short of realizing dual superconformal symmetry in the string theory dual [19,20]. Dual superconformal symmetry would imply a Yangian symmetry, although the converse does not hold. Further studies are required to see whether the Yangian symmetry of SCS_6 extends to all tree-level amplitudes and, if so, whether it originates from some dual superconformal symmetry.

The aim of this Letter is to provide further support for the relevance of Yangian symmetry in SCS_6. Generalizing the approach of [3], we present a generating function for scattering amplitudes of $SCS_6 - \mathcal{L}_{2k}$ in Eq. (8)—in the same sense as $\mathcal{L}_{n,k}$ of SYM_4 and give a formal proof of its Yangian invariance for all k.

We begin with a quick review of Witten's twistor formulation [21] on which $\mathcal{L}_{n,k}$ of [3] is based, and explain how it should be generalized to three dimensions. Using a supersymmetric version of the three dimensional twistor, we write down the generating function $\mathcal{L}_{2k}$ for SCS_6 and study its properties. We verify superconformal invariance, cyclic symmetry and Yangian invariance, and also show that it reproduces some known tree-level amplitudes. We conclude with a discussion on dual superconformal symmetry and directions for future works.

Twistor in four dimensions vs three dimensions.—In four dimensions, a null momentum can be written as a bi-spinor $p^{\alpha\dot{\alpha}} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$. A standard way to introduce twistors [21] is to take a Fourier transform of the plane wave e^{ipx} with respect to one of the two spinors:

$$\int e^{ix_{\alpha\dot{\alpha}}\lambda^\alpha\bar{\lambda}^{\dot{\alpha}}} e^{-i\bar{\mu}_\alpha\lambda^\alpha} d^2\lambda \propto \delta(\bar{\mu}_\alpha - x_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}). \quad (1)$$

The δ function enforces the defining equation for the twistor variables $(\bar{\mu}_\alpha, \bar{\lambda}^{\dot{\alpha}})$. Equivalently, we can regard $\bar{\mu}_\alpha = -i(\partial/\partial\lambda^\alpha)$ as a "momentum operator" acting on "wave functions" in the $(\lambda, \bar{\lambda})$ space and reinterpret the twistor equation as a wave equation,

$$(\bar{\mu}_\alpha - x_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}) \exp(ix_{\alpha\dot{\alpha}}\lambda^\alpha\bar{\lambda}^{\dot{\alpha}}) = 0. \quad (2)$$

In three dimensions, the bi-spinor notation involves a single spinor, $p^{\alpha\beta} = \lambda^\alpha\lambda^\beta$. Introducing the operator $\mu_\alpha = i(\partial/\partial\lambda^\alpha)$, we can again realize the three dimensional twistor equation [22,23] as a wave equation:

$$(\mu_\alpha - x_{\alpha\beta}\lambda^\beta) \exp\left(-\frac{i}{2} x_{\alpha\beta}\lambda^\alpha\lambda^\beta\right) = 0. \quad (3)$$

Drawing an analogy from quantum mechanics, we note that the n-point amplitude can be treated as a wave function in $2n$ "coordinate" variables $\{\lambda_i^\alpha\}$ $(i = 1, \ldots, n)$, whereas the conformal symmetry $SO(2,3) \simeq Sp(4,\mathbb{R})$ acts linearly on the $4n$ dimensional "phase space" parametrized by $\{Z_i^A = (\lambda_i^\alpha, \mu_{i\alpha})\}$.

Supertwistor.—The on-shell superfield for SCS_6 involves three fermionic coordinates η^I in addition to λ^α [17]. The particle and antiparticle superfields take the form

$$\Phi = \phi^4 + \eta^I \psi_I + \tfrac{1}{2}\epsilon_{IJK}\eta^I\eta^J\phi^K + \tfrac{1}{6}\epsilon_{IJK}\eta^I\eta^J\eta^K\psi_4,$$
$$\bar{\Phi} = \bar{\psi}^4 + \eta^I \bar{\phi}_I + \tfrac{1}{2}\epsilon_{IJK}\eta^I\eta^J\bar{\psi}^K + \tfrac{1}{6}\epsilon_{IJK}\eta^I\eta^J\eta^K\bar{\phi}_4, \tag{4}$$

where the scalars ϕ and fermions ψ are all understood as functions of the momentum spinor λ. The $SO(6)$ R symmetry acting on the $\mathcal{N} = 6$ supercharges are realized by η^I and their conjugates $\zeta_I = \partial/\partial\eta^I$ through the oscillator algebra,

$$\{\eta^I, \zeta_J\} = \delta^I_J \qquad (I, J = 1, 2, 3). \tag{5}$$

Note that the quantum mechanics analogy introduced above remains valid even after including fermions; while the superamplitude $\mathcal{A}(\Lambda)$ can be regarded as a wave function in the "half supertwistor" variables $\Lambda_i = (\lambda^\alpha, \eta^I)_i$, the generators of the full superconformal symmetry $OSp(6|4)$ are represented by quadratic products of the "full supertwistor" variables

$$\mathcal{Z}_i^{\mathcal{A}} = (\lambda^\alpha, \mu_\alpha; \eta^I, \zeta_I)_i \sim (\Lambda, \partial/\partial\Lambda)_i \tag{6}$$

to be interpreted as operators acting on $\mathcal{A}(\Lambda)$. In this formulation, only the $U(1,1|3) \subset OSp(6|4)$ acts linearly on Λ with generators of the form $(\Lambda\partial/\partial\Lambda)$. The rest of the generators act, schematically, either as multiplications $(\Lambda\Lambda)$ or as second order derivatives $(\partial^2/\partial\Lambda\partial\Lambda)$. Although both λ and η are subject to reality conditions, we will loosely treat them as complex variables $\Lambda \in \mathbb{C}^{2|3}$ in what follows.

Amplitudes of SCS_6.—The superfields Φ and $\bar{\Phi}$ in (4) transform in mutually complex conjugate representations of the gauge group; a prime example is $U(N) \times U(N)$ gauge group with Φ transforming in $(\mathbf{N}, \bar{\mathbf{N}})$ and $\bar{\Phi}$ in $(\bar{\mathbf{N}}, \mathbf{N})$. Barring the possibility of a "baryonic" vertex such as $\det(\Phi)$ which scale as Φ^N, the nonvanishing amplitudes must carry equal number of Φ and $\bar{\Phi}$. Moreover, one can define color-ordered amplitudes such that the external legs alternate between Φ and $\bar{\Phi}$ [17]. In summary, we are interested in the $n(= 2k)$-point color-ordered superamplitudes

$$\mathcal{A}_{n=2k}(\Lambda) = \mathcal{A}_{2k}(\Lambda_1, \Lambda_2, \ldots, \Lambda_{2k}), \tag{7}$$

where by convention we associate $\Lambda_{\text{odd(even)}}$ to $\bar{\Phi}(\Phi)$ (opposite to the convention of [17]). Because Φ and $\bar{\Phi}$ carry opposite statistics, $\mathcal{A}_{2k}$ acquires a factor of $(-1)^{k-1}$ upon cyclic permutation by two sites [17]. The component amplitudes can be read off from the superamplitude as the coefficients of various monomials of η_i^I. They are rational functions of Lorentz invariant products of the momentum spinors, $\langle ij\rangle \equiv \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta$.

The generating function.—Our proposal for the generating function for the $n(= 2k)$-point amplitude is

$$\mathcal{L}_{2k}(\Lambda) = \int \frac{d^{k\times 2k}C}{\text{vol}[GL(k)]} \frac{\delta^{(k(k+1)/2)}(CC^T)\delta^{2k|3k}(C\Lambda)}{M_1 M_2 \cdots M_k}. \tag{8}$$

This form of $\mathcal{L}_{2k}$ was partly motivated by the formal similarity noted in [17] between $\mathcal{A}_{2k}$ of SCS_6 and $\mathcal{A}_{2k,k}$ of SYM_4. As will become gradually clearer, both the similarity and the difference between $\mathcal{L}_{n,k}$ of [3] and $\mathcal{L}_{2k}$ here can be traced back to the structure of the momentum spinor: $p^{\alpha\dot{\alpha}} = \lambda^\alpha\bar{\lambda}^{\dot{\alpha}}$ in four dimensions and $p^{\alpha\beta} = \lambda^\alpha\lambda^\beta$ in three dimensions.

The integration variable C is a $(k \times 2k)$ matrix. The dot products denote $(CC^T)_{mn} = C_{mi}C_{ni}$, $(C\Lambda)_m = C_{mi}\Lambda_i$. M_i represents the ith minor of C defined by

$$M_i = \epsilon^{m_1\cdots m_k} C_{m_1(i)} C_{m_2(i+1)} \cdots C_{m_k(i+k-1)}. \tag{9}$$

The measure $d^{k\times 2k}C$ is covariant under a $GL(k) \times GL(2k)$ group action on the left/right. The $\text{vol}[GL(k)]^{-1}$ factor is a reminder that the $GL(k)$-left action is an exact symmetry of the integrand and should be "gauge fixed." The $GL(2k)$-right would-be symmetry is reduced to $O(2k)$ by $\delta(CC^T)$, which is in turn broken spontaneously by $\delta(C\Lambda)$ and explicitly by the denominator.

The cyclic symmetry of (8) is obscured by the presence of only k out of $2k$ minors. However, one can use the constraint $CC^T = 0$ to show that

$$M_i M_{i+1} = (-1)^{k-1} M_{i+k} M_{i+1+k}. \tag{10}$$

Thus $\mathcal{L}_{2k}$ transforms in the same way as $\mathcal{A}_{2k}$ under cyclic permutation by two sites.

The net number of integration variables in (8) can be counted as follows (cf. [3]). Starting from $2k^2$ elements of C, subtracting k^2 for the $GL(k)$ gauge fixing and $2k$ for the bosonic δ functions, and pulling out the overall momentum conserving δ function, we are left with

$$2k^2 - k^2 - \frac{k(k+1)}{2} - 2k + 3 = \frac{(k-2)(k-3)}{2}. \tag{11}$$

Superconformal invariance.—To begin with, note that $\mathcal{L}_{2k}$ has degree $-2k$ in λ and $+3k$ in η in agreement with the degree counting for $\mathcal{A}_{2k}$ from Feynman diagrams [17]. To verify superconformal invariance in the half supertwistor notation, we need to consider three cases separately. The $\delta(C\Lambda)$ factor is manifestly invariant under the linearly realized $U(1,1|3)$ subgroup. The two-derivative generators $(\partial^2/\partial\Lambda\partial\Lambda)$ acting on $\delta^{2k|3k}(C\Lambda)$ produces CC^T to be annihilated by $\delta(CC^T)$. To see the invariance under the multiplication generators ($\Lambda\Lambda$), note that the constraint $CC^T = 0$ generically defines k linearly independent null vectors in $\mathbb{C}^{2k}$. One can construct another ($k \times 2k$) matrix $\hat{C}$ composed of k dual null vectors in $\mathbb{C}^{2k}$ satisfying

$$\hat{C} \cdot \hat{C}^T = 0, \qquad C \cdot \hat{C}^T = I_{k\times k}, \tag{12}$$

and use the completeness relation for C and $\hat{C}$ to write

$$\Lambda^T \cdot \Lambda = \Lambda^T \cdot (C^T \hat{C} + \hat{C}^T C) \cdot \Lambda, \tag{13}$$

which is annihilated by the $\delta(C\Lambda)$ factor.

Reproducing known amplitudes.—Only 4- and 6-point amplitudes of SCS_6 are available in the literature so far [16–18]. In both cases, it is straightforward to show that $\mathcal{L}_{2k}$ reproduces the amplitudes mainly because there is no integral to do according to the counting (11).

Here, we only discuss the 4-point amplitude and refer the reader to [24] for the 6-point result. For simplicity, we begin with the 4-scalar component amplitude [17],

$$A_{4\phi} = \frac{\langle 13\rangle^3}{\langle 14\rangle\langle 43\rangle}\delta^{(3)}(\lambda_r\lambda_r + \lambda_{\bar{s}}\lambda_{\bar{s}}), \tag{14}$$

where we divided the particle indices $\{i = 1, \ldots, 4\}$ into $\{r = 1, 3\}$ and $\{\bar{s} = 2, 4\}$. Consider the partial Fourier transform,

$$\hat{A}_{4\phi}(\lambda_r, \mu_{\bar{s}}) = \int A_{4\phi}(\lambda_r, \lambda_{\bar{s}})e^{-i\mu_{\bar{s}}\lambda_{\bar{s}}}d^4\lambda_{\bar{s}}, \tag{15}$$

and introduce the "link matrices" (cf. [25]) defined by

$$\lambda_{\bar{s}} = -c_{r\bar{s}}\lambda_r. \tag{16}$$

After the change of variable from $\lambda_{\bar{s}}^\alpha$ to $c_{r\bar{s}}$, we obtain (up to an overall coefficient)

$$\hat{A}_{4\phi}(\lambda_r, \mu_{\bar{s}}) = \int \frac{d^4c_{r\bar{s}}\delta^{(3)}(\delta_{rp} + c_{r\bar{s}}c_{p\bar{s}})}{c_{14}c_{34}}e^{ic_{r\bar{s}}\lambda_r\mu_{\bar{s}}}. \tag{17}$$

Taking the inverse Fourier transform back to $A_{4\phi}(\lambda_r, \lambda_{\bar{s}})$ and reinstating the fermions, we recognize the final result as a gauge-fixed version of $\mathcal{L}_{2k}$ with

$$C = \begin{pmatrix} c_{12} & 1 & c_{32} & 0 \\ c_{14} & 0 & c_{34} & 1 \end{pmatrix}. \tag{18}$$

Integrability via Yangian symmetry.—The original superconformal invariance alone is far from sufficient to determine the amplitude uniquely. For instance, we can multiply $\mathcal{L}_{2k}$ by an arbitrary function $f(\mathcal{Z}_i\mathcal{Z}_j)$ without breaking superconformal invariance, where the product $\mathcal{Z}_i\mathcal{Z}_j$ is defined by the OSp(6|4)-invariant metric.

In four dimensions, under mild assumptions, $\mathcal{L}_{n,k}$ was proven to be the unique Yangian invariants [26–28]. Encouraged by the Yangian invariance of 4- and 6-point amplitudes [17] and the fact that $\mathcal{L}_{2k}$ reproduces them, we now move on to examine the Yangian invariance of $\mathcal{L}_{2k}$ for all k. We will show that $\mathcal{L}_{2k}$ is annihilated by the level one Yangian generators, which together with the superconformal invariance guarantees the full Yangian invariance [10]. The uniqueness problem is left for a future work.

We mostly follow the methods developed in [26] to prove the Yangian invariance of $\mathcal{L}_{n,k}$. As shown in [10,17], the level one Yangian generators can be written in the bilinear form,

$$\mathcal{J}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j}(-1)^{\mathcal{C}}(J_i{}^{\mathcal{A}}{}_{\mathcal{C}}J_j{}^{\mathcal{C}}{}_{\mathcal{B}} - J_j{}^{\mathcal{A}}{}_{\mathcal{C}}J_i{}^{\mathcal{C}}{}_{\mathcal{B}}), \tag{19}$$

where $J_i{}^{\mathcal{A}}{}_{\mathcal{B}}$ are the superconformal generators acting on the ith particle. In terms of the full supertwistors $\mathcal{Z}_i^{\mathcal{A}}$, the generators can be written as

$$\mathcal{J}^{\mathcal{A}}{}_{\mathcal{B}} = \sum_{i<j}[\mathcal{Z}_i^{\mathcal{A}}\mathcal{Z}_{\mathcal{B}j}\mathcal{Z}_i^{\mathcal{C}}\mathcal{Z}_{\mathcal{C}j} - i\mathcal{Z}_i^{\mathcal{A}}\mathcal{Z}_{\mathcal{B}i} - (i \leftrightarrow j)]. \tag{20}$$

The key insight we adopt from [26] is that $\mathcal{Z}_i^{\mathcal{C}}\mathcal{Z}_{\mathcal{C}j}$ generates an $O(2k)$ action on $\{\Lambda_i\}$. Using the covariance of $\delta^{2k|3k}(C\Lambda)$, we can trade it with an inverse $O(2k)$ action on the matrix C. In other words, we can replace $\mathcal{Z}_i^{\mathcal{C}}\mathcal{Z}_{\mathcal{C}j}$ by $O_{ij} = i\sum_{m=1}^k(C_{mi}\partial/\partial C_{mj} - C_{mj}\partial/\partial C_{mi})$. The factors dC and $\delta(CC^T)$ are invariant under the $O(2k)$ action, so we can do an integration by parts to make O_{ij} act on the denominator. Then, following essentially the same steps as in [26], we can show that the quartic and quadratic terms in (20) acting on $\mathcal{L}_{2k}$ cancel each other.

Dual superconformal symmetry and momentum-twistor.—In four dimensions, there is an alternative way to prove the Yangian invariance directly through its relation to dual superconformal symmetry [10]. It was shown in [29] that after a suitable change of variables, $\mathcal{L}_{n,k}$ can be rewritten as $\mathcal{L}_{n,k} = \mathcal{A}_n^{\mathrm{MHV}}\mathcal{R}_{n,k}$, where $\mathcal{A}_n^{\mathrm{MHV}}$ is the n-point maximally helicity violating (MHV) amplitude and $\mathcal{R}_{n,k}$ is another integral formula with manifest dual superconformal invariance [30]. The change of variable to go from $\mathcal{L}_{n,k}$ to $\mathcal{R}_{n,k}$ can be interpreted in terms of the "momentum twistor" introduced in [31].

In our case, it is straightforward to make a change of variable similar to that of [29] to obtain

$$\mathcal{L}_{2k} = \frac{\delta^3(P)\delta^6(Q)}{[\langle 12\rangle\cdots\langle 2k-12k\rangle\langle 2k1\rangle]^{1/2}}\mathcal{R}_{2k}. \tag{21}$$

This factorization was noted previously in [17]. For the bosonic variables, the notion of momentum twistor continues to hold with little modification. But, we have not found a satisfactory interpretation for the new fermionic coordinates that enter $\mathcal{R}_{2k}$ in terms of a "momentum supertwistor."

The relation between dual superconformal symmetry and Yangian symmetry in SCS_6 has been elucidated in a very recent paper [32]. The dual superconformal generators are naturally defined in the "dual space" (cf. [7]),

$$x_i^{\alpha\beta} - x_{i+1}^{\alpha\beta} = \lambda_i^\alpha\lambda_i^\beta, \qquad \theta_i^{I\alpha} - \theta_{i+1}^{I\alpha} = \lambda_i^\alpha\eta_i^I. \tag{22}$$

According to [32], dual superconformal symmetry of SCS_6 requires some additional bosonic coordinates in the dual space, which account for the missing piece in previous search [19,20] for dual superconformal symmetry via "fermionic T duality" in string theory [8,9]. How the results of [32] may relate to the momentum-supertwistor is an interesting open problem.

Outlook.—Although the results of this Letter are quite suggestive, much work remains to be done to establish the connection between the generating function $\mathcal{L}_{2k}$ and the amplitudes $\mathcal{A}_{2k}$ to the same extent as their four dimensional counterparts. Among other things, a precise prescription for the integration contour will be needed to deduce recursion relations analogous to [33,34] from $\mathcal{L}_{2k}$, which in turn could be related to the usual perturbation theory in terms of Feynman diagrams. A completely rigorous proof of Yangian invariance [35] taking account of anomalies in collinear limits [36] will also rely on the correct choice of contours. A variant of the geometric picture based on Grassmannian explained in [3] will be very useful in solving the contour problem. Some of these issues are currently under investigation and will be reported in [24].

We thank the organizers of the Cargese Summer School on String Theory: Formal Developments and Applications, where the work was initiated with inspiration from the lectures of N. Arkani-Hamed and Z. Bern. We are especially indebted to A. Lipstein and Y.-t. Huang for many illuminating discussions and for sharing their work [32] prior to publication. We also benefitted from discussions with D. Bak and H. Johansson. This work was supported in part by National Research Foundation of Korea (NRF) Grants No. 2007-331-C00073, 2009-0072755, and 2009-0084601.

[1] Z. Bern, L. J. Dixon, and D. A. Kosower, Ann. Phys. (Leipzig) **322**, 1587 (2007).
[2] M. Wolf, arXiv:1001.3871.
[3] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, J. High Energy Phys. 03 (2010) 020.
[4] J. L. Bourjaily, J. Trnka, A. Volovich, and C. Wen, arXiv:1006.1899.
[5] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot, and J. Trnka, arXiv:1008.2958.
[6] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 06 (2007) 064.
[7] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B828**, 317 (2010).
[8] N. Berkovits and J. Maldacena, J. High Energy Phys. 09 (2008) 062.
[9] N. Beisert, R. Ricci, A. A. Tseytlin, and M. Wolf, Phys. Rev. D **78**, 126004 (2008).
[10] J. M. Drummond, J. M. Henn, and J. Plefka, J. High Energy Phys. 05 (2009) 046.
[11] N. Beisert, arXiv:1004.5423.
[12] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (2008) 091.
[13] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, and J. Park, J. High Energy Phys. 07 (2008) 091.
[14] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, and J. Park, J. High Energy Phys. 09 (2008) 002.
[15] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[16] A. Agarwal, N. Beisert, and T. McLoughlin, J. High Energy Phys. 06 (2009) 045.
[17] T. Bargheer, F. Loebbert, and C. Meneghelli, Phys. Rev. D **82**, 045016 (2010).
[18] Y.-t. Huang and A. Lipstein, arXiv:1004.4735.
[19] I. Adam, A. Dekel, Y. Oz, J. High Energy Phys. 04 (2009) 120.
[20] P. A. Grassi, D. Sorokin, L. Wulff, J. High Energy Phys. 08 (2009) 060.
[21] E. Witten, Commun. Math. Phys. **252**, 189 (2004).
[22] N. J. Hitchin, Commun. Math. Phys. **83**, 579 (1982).
[23] D.-W. Chiou, O. J. Ganor, Y. P. Hong, B. S. Kim, and I. Mitra, Phys. Rev. D **71**, 125016 (2005).
[24] Work in progress.
[25] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, J. High Energy Phys. 03 (2010) 110.
[26] J. M. Drummond and L. Ferro, J. High Energy Phys. 07 (2010) 027.
[27] J. M. Drummond and L. Ferro, arXiv:1002.4622.
[28] G. P. Korchemsky and E. Sokatchev, arXiv:1002.4625.
[29] N. Arkani-Hamed, F. Cachazo, and C. Cheung, J. High Energy Phys. 03 (2010) 036.
[30] L. Mason and D. Skinner, J. High Energy Phys. 11 (2009) 045.
[31] A. Hodges, arXiv:0905.1473.
[32] Y.-t. Huang and A. E. Lipstein, arXiv:1008.0041.
[33] F. Cachazo, P. Svrcek, and E. Witten, J. High Energy Phys. 09 (2004) 006.
[34] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005).
[35] T. Bargheer, N. Beisert, W. Galleas, F. Loebbert, and T. McLoughlin, J. High Energy Phys. 11 (2009) 056.
[36] F. Cachazo, P. Svrcek, and E. Witten, J. High Energy Phys. 10 (2004) 077.