Efficiency at Maximum Power of Low-Dissipation Carnot Engines

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We study the efficiency at maximum power, η^*, of engines performing finite-time Carnot cycles between a hot and a cold reservoir at temperatures T_h and T_c, respectively. For engines reaching Carnot efficiency $\eta_C = 1 - T_c/T_h$ in the reversible limit (long cycle time, zero dissipation), we find in the limit of low dissipation that η^* is bounded from above by $\eta_C/(2 - \eta_C)$ and from below by $\eta_C/2$. These bounds are reached when the ratio of the dissipation during the cold and hot isothermal phases tend, respectively, to zero or infinity. For symmetric dissipation (ratio one) the Curzon-Ahlborn efficiency $\eta_{CA} = 1 - \sqrt{T_c/T_h}$ is recovered.

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Thermal machines performing Carnot cycles transform a certain amount of heat Q_h from a hot reservoir at temperature T_h into an amount of work $-W$, with the remaining energy being evacuated as heat $Q_c = -Q_h - W$ to a cold reservoir at temperature T_c. We adopted here the usual convention that heat and work absorbed by the system are positive. By assuming that there is no perpetuum mobile of the second kind, more precisely that heat does not spontaneously flow from a cold to a hot reservoir, Carnot was able to show that the efficiency of the heat-work transformation

$$\eta = -\frac{W}{Q_h} = 1 + \frac{Q_c}{Q_h} \tag{1}$$

is universally bounded by a maximum value, the so-called Carnot efficiency

$$\eta_C = 1 - \frac{T_c}{T_h}. \tag{2}$$

This insight lies at the heart of thermodynamics, since it led Clausius to the introduction of the entropy, the state function which is central for the formulation of the Second Law. The entropy change of a system is given by $\Delta S = \int_{qs} \bar{d}Q/T$, where the integral is over the infinitesimal amounts of absorbed heat $\bar{d}Q$ for a quasistatic transformation of the system (i.e., a succession of equilibrium states) [1]. The total entropy production during a Carnot cycle is given by

$$\Delta S_{\rm tot} = -\frac{Q_h}{T_h} - \frac{Q_c}{T_c} = (\eta_C - \eta)\frac{Q_h}{T_c} \tag{3}$$

since the auxiliary work-performing system returns to its initial state (hence no change in its entropy $\Delta S = 0$) and the heat reservoirs are assumed to undergo a quasistatic heat exchange while preserving their temperature. The fact that the total entropy cannot decrease, $\Delta S_{\rm tot} \geq 0$, is equivalent to the statement that efficiency is bounded by Carnot efficiency, $\eta \leq \eta_C$. The latter is reached for a reversible process, $\Delta S_{\rm tot} = 0$, which can only be achieved for a quasistatic transformation of the system implying infinitely slow Carnot cycles.

While the concept of Carnot efficiency is of paramount importance in the derivation of thermodynamics, its practical implications are more limited: to reach the reversible limit, one needs in principle to work with infinitely slow cycles; hence, the power of such a thermal machine is zero. This leaves open the question of efficiency at finite power. Although this issue was first addressed by Chambadal [2] and Novikov [3], it is often associated with the later work of Curzon and Ahlborn (CA) [4]. Using an approximate analysis of a finite-time Carnot cycle, they observed that the power goes through a maximum, and that the corresponding efficiency at maximum power is given by the appealing expression

$$\eta_{\rm CA} = 1 - \sqrt{\frac{T_c}{T_h}}. \tag{4}$$

Unfortunately, the CA efficiency turns out to be neither an exact nor a universal result, and it is neither an upper nor a lower bound [5]. Yet it describes the efficiency of actual thermal plants very well [1,4,6], and is reasonably close to the efficiency at maximum power for several model systems [7–18]. How does this agreement come about?

As a first explanation, we note that the underlying time-reversibility of the laws of physics under some conditions implies universal properties for the efficiency at maximum power. More precisely, let us consider the expansion of the efficiency at maximum power in terms of the Carnot efficiency η_C. For CA efficiency, one has $\eta_{\rm CA} = 1 - \sqrt{1-\eta_C} = \eta_C/2 + \eta_C^2/8 + \cdots$. It was proven from the symmetry of the Onsager coefficients that the coefficient $1/2$ is actually an upper bound for the linear response at maximum power, and that the bound is reached for strong coupling between the heat-performing and the work-performing fluxes [19]. Using the equivalent of Onsager symmetry at the level of nonlinear response, one can show that the coefficient of η_C^2 is also universal, i.e., equal to $1/8$, for strongly coupled fluxes possessing in addition a left-right symmetry [20].

In this Letter, we further clarify the special status of CA efficiency: it turns out to be an exact property for Carnot machines operating under conditions of low, symmetric dissipation. The argument is very simple, as can be expected from its claim of generality. Our starting point is a Carnot engine which operates under reversible conditions when the durations of the cycles become very large, i.e., when the system always remains infinitesimally close to equilibrium all along the cycle. While in contact with the hot reservoir, the work-performing auxiliary system absorbs an amount of heat Q_h, resulting in a system entropy change $\Delta S = Q_h/T_h$. During the heat exchange with the cold reservoir, the entropy of the system returns to its original value, decreasing by an amount $-\Delta S = Q_c/T_c$. From the equality $Q_h/T_h = -Q_c/T_c$ we recover Carnot efficiency $\eta = 1 + Q_c/Q_h = 1 - T_c/T_h$. We next consider finite-time cycles which move the engine away from the reversible regime.

Let τ_c (τ_h) be the time durations during which the system is in contact with the cold (hot) reservoir along a cycle. In the weak dissipation regime, the system relaxation is assumed to be fast compared to τ_h and τ_c. The entropy production per cycle along the cold (hot) part of the cycle is expected to behave as Σ_c/τ_c (Σ_h/τ_h) since the reversible regime is approached in the limits $\tau_h \to \infty$ and $\tau_c \to \infty$ (for a further comment on this assumption, see [21]). As a result, the amount of heat per cycle entering the system from the cold (hot) reservoir will be

$$Q_c = T_c\left(-\Delta S - \frac{\Sigma_c}{\tau_c} + \ldots\right); \qquad Q_h = T_h\left(\Delta S - \frac{\Sigma_h}{\tau_h} + \ldots\right). \tag{5}$$

Note that we did not specify the details of the procedure by which we deviate from the reversible scenario. This information is contained in the coefficients Σ_c and Σ_h. They express how dissipation increases as one moves away from the reversible limit. We also do not need to assume that the temperature difference between T_c and T_h is small; hence the expansion is not limited to the linear response regime.

We now consider the power generated during this Carnot cycle. Using (5), we get

$$P = \frac{-W}{\tau_h + \tau_c} = \frac{Q_h + Q_c}{\tau_h + \tau_c} = \frac{(T_h - T_c)\Delta S - T_h\Sigma_h/\tau_h - T_c\Sigma_c/\tau_c}{\tau_h + \tau_c}. \tag{6}$$

The maximum power is found by setting the derivatives of P with respect to τ_h and τ_c equal to zero. We find a unique physically acceptable solution at

$$\tau_h = 2\frac{T_h\Sigma_h}{(T_h - T_c)\Delta S}\left(1 + \sqrt{\frac{T_c\Sigma_c}{T_h\Sigma_h}}\right)\tau_c = 2\frac{T_c\Sigma_c}{(T_h - T_c)\Delta S}\left(1 + \sqrt{\frac{T_h\Sigma_h}{T_c\Sigma_c}}\right). \tag{7}$$

Using (5) with (7) in the efficiency (1) leads to the main result of this paper, namely, the following expression for the efficiency at maximum power:

$$\eta^* = \frac{\eta_C\left(1 + \sqrt{\frac{T_c\Sigma_c}{T_h\Sigma_h}}\right)}{\left(1 + \sqrt{\frac{T_c\Sigma_c}{T_h\Sigma_h}}\right)^2 + \frac{T_c}{T_h}\left(1 - \frac{\Sigma_c}{\Sigma_h}\right)}. \tag{8}$$

This result was previously obtained by Schmiedl and Seifert using a Fokker-Plank formulation of stochastic thermodynamics [10]. We present it here in a broader context by arguing that the expansion (5) is generic in the weak dissipation limit if a reversible long time limit exists. For symmetric dissipation, $\Sigma_h = \Sigma_c$, we recover the Curzon-Ahlborn efficiency:

$$\eta^* = \eta_{\rm CA} = 1 - \sqrt{\frac{T_c}{T_h}} = 1 - \sqrt{1 - \eta_C}. \tag{9}$$

We also note that (8) can be expanded in η_C as

$$\eta^* = \frac{\eta_C}{2} + \frac{\eta_C^2}{4 + 4\sqrt{\Sigma_c/\Sigma_h}} + \mathcal{O}(\eta_C^3) \tag{10}$$

The coefficient of the second order term lies between 0 and $1/4$ and for symmetric dissipation we recover the $1/8$, as discussed in [20]. Symmetric dissipation for time-dependent cycles is thus similar to the left-right symmetry on the fluxes [see Eq. (20) of Ref. [20]] which is required to recover the universal value of the quadratic coefficient for steady-state problems.

We now turn to the main focus of the result (8). In the limits $\Sigma_c/\Sigma_h \to 0$ and $\Sigma_c/\Sigma_h \to \infty$, the efficiency at

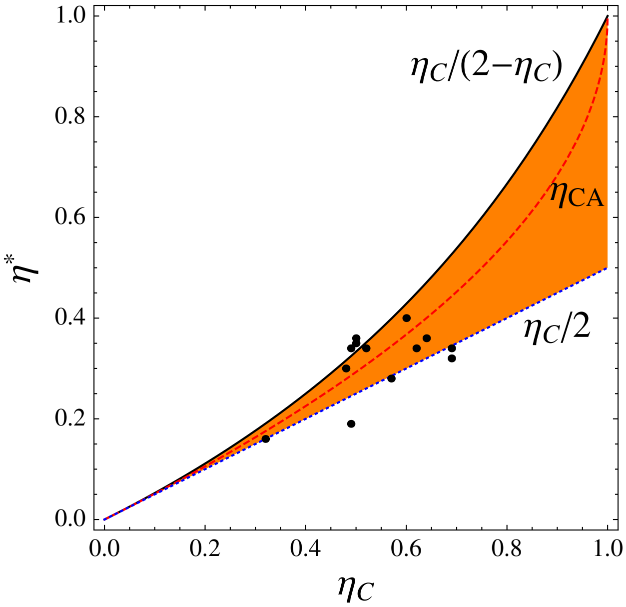

FIG. 1 (color online). Efficiency at maximum power as a function of η_C. The upper and lower bounds of the efficiency given by Eq. (11) are denoted by a black solid line and a (blue) dotted line, respectively. The Curzon-Ahlborn efficiency is the (red) dashed line. The dots represent the observed efficiencies of the various thermal power plants reported in Table I. Observed efficiencies above and below the bounds could result from power plants not operating at maximum power.

maximum power converges to the upper bound $\eta_+ = \eta_C/(2-\eta_C)$ and to the lower bound $\eta_- = \eta_C/2$, respectively:

$$\frac{\eta_C}{2} \equiv \eta_- \leq \eta^* \leq \eta_+ \equiv \frac{\eta_C}{2-\eta_C}. \qquad (11)$$

In Fig. 1 we plot the efficiency (8) as a function of η_C comparing the CA result with the upper and lower bounds (11). We note that these bounds were previously derived by assuming a specific form of heat transfers in [25]. The upper bound η_+, which is reached in the completely asymmetric limit $\Sigma_c/\Sigma_h \to 0$, is particularly interesting. It coincides with a reported universal upper bound that was derived in [23] [cf. Eq. (16)] using a very different approach. It also agrees with the upper bound obtained by optimizing with respect to the temperature of the hot reservoir [24]. Finally, it also arises in a model for the Feynman ratchet [26] [cf. Eq. (25)].

In order to identify the regime of operation of a particular engine other than via the ratio of coefficients Σ_c/Σ_h, we evaluate the ratio of the contact times at maximum power:

$$\frac{\tau_c}{\tau_h} = \sqrt{\frac{T_c\Sigma_c}{T_h\Sigma_h}}. \qquad (12)$$

We conclude that symmetric dissipation corresponds to the case when this ratio is equal to the square root of the ratio of the temperatures,

$$\frac{\tau_c}{\tau_h} = \sqrt{\frac{T_c}{T_h}}, \qquad (13)$$

whereas maximum and minimum efficiency are reached for the highly asymmetric cases

$$\frac{\tau_c}{\tau_h} \to 0 \quad \text{and} \quad \frac{\tau_c}{\tau_h} \to \infty. \qquad (14)$$

In conclusion, we have presented a simple and general argument for estimating efficiency of a thermal engine at maximum power. The main bonuses of this analysis are the derivation of the Curzon-Ahlborn efficiency in the case of symmetric dissipation, and the prediction of an upper and lower bound reached in the limits of extremely asymmetric dissipation. While actual plants usually operate under steady-state conditions rather than as a Carnot cycle, and

TABLE I. Theoretical bounds and observed efficiency η_{obs} of thermal plants.

Plant	$T_h(K)$	$T_c(K)$	η_C	η_-	η_+	η_{obs}
Doel 4 (Nuclear, Belgium) [6]	566	283	0.5	0.25	0.33	0.35
Almaraz II (Nuclear, Spain) [6]	600	290	0.52	0.26	0.35	0.34
Sizewell B (Nuclear, UK) [6]	581	288	0.5	0.25	0.34	0.36
Cofrentes (Nuclear, Spain) [6]	562	289	0.49	0.24	0.32	0.34
Heysham (Nuclear, UK) [6]	727	288	0.60	0.30	0.43	0.40
West Thurrock (Coal, UK) [1]	838	298	0.64	0.32	0.48	0.36
CANDU (Nuclear, Canada) [1]	573	298	0.48	0.24	0.32	0.30
Larderello (Geothermal, Italy)[1]	523	353	0.32	0.16	0.19	0.16
Calder Hall (Nuclear, UK) [6]	583	298	0.49	0.24	0.32	0.19
(Steam/Mercury,USA) [6]	783	298	0.62	0.31	0.45	0.34
(Steam, UK) [6]	698	298	0.57	0.29	0.40	0.28
(Gas Turbine, Switzerland) [6]	963	298	0.69	0.35	0.53	0.32
(Gas Turbine, France) [6]	953	298	0.69	0.34	0.52	0.34

while the assumptions of low dissipation and maximum power may not hold, one feels compelled to compare the upper and lower bounds with observed efficiencies, as is done in Table I and in Fig. 1.

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[21] In general, reconnecting the system to one of the reservoirs will induce dissipation because the system is not in equilibrium with this reservoir. This leads to an additional contribution to the entropy production in (5), which can have a complicated time dependence and does not vanish in the very slow cycle limit. This unavoidably results in a decrease of the efficiency at maximum power. By considering cycles with a reversible limit, we assume this dissipation to be negligible or absent. This assumption is reasonable in large systems but might require a large number of external control parameters for small systems (see, for example, [22]). One can then use a calculation similar to the adiabatic theory in quantum mechanics to show that the entropy production related to the change of the system during the times τ while in contact with the reservoir behaves as $1/\tau$.
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