Dynamical Conductivity at the Dirty Superconductor-Metal Quantum Phase Transition

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We study the transport properties of ultrathin disordered nanowires in the neighborhood of the superconductor-metal quantum phase transition. To this end we combine numerical calculations with analytical strong-disorder renormalization group results. The quantum critical conductivity at zero temperature diverges logarithmically as a function of frequency. In the metallic phase, it obeys activated scaling associated with an infinite-randomness quantum critical point. We extend the scaling theory to higher dimensions and discuss implications for experiments.

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Electrical transport in low-dimensional strongly fluctuating superconductors has been the subject of intense experimental investigation for almost half a century [1]. Recently, advances in experimental techniques have allowed for the fabrication of ultrathin metallic nanowires having diameters smaller than the bulk superconducting coherence length, but large enough to include many transverse channels for electronic conduction. Resistance measurements have shown that the thicker among these wires exhibit a well-defined phase transition from a resistive to a superconducting state with decreasing temperature, while thinner wires appear to remain resistive down to the lowest temperatures measured [2–6].

It has been proposed [7–10] that these experiments may be understood in terms of a superconductor-metal quantum phase transition (SMT) driven by pair-breaking interactions, possibly due to random magnetic moments trapped on the wire surface [11]. A description of this transition is provided by a theory, first proposed by Feigel'man and Larkin [12], of a complex Cooper pair order parameter whose fluctuations are damped by decay into unpaired electrons [13–17].

As the nanowires are prone to random variations in diameter and because of the random positions of the pair-breaking moments, quenched disorder plays an important role. The thermodynamics of the disordered SMT has been analyzed both analytically [18] and numerically [10] in the relevant case of one space dimension. It is governed, for any nonzero disorder strength, by a nonperturbative infinite-randomness critical point (IRCP). This IRCP is in the same universality class as the magnetic quantum critical point of the random transverse-field Ising chain despite the fact that the two systems have different symmetries: The clean transverse-field Ising chain can be described by relativistic free fermions (and, therefore, dynamical exponent $z = 1$) whereas the clean SMT is described by overdamped O(2) fluctuations with $z = 2$. The homology lies in the marginal dynamics of finite size clusters in both models [19] which are the famous *rare regions* of Griffiths-McCoy physics [20].

Many asymptotically exact results for the random transverse-field Ising chain [21,22] apply directly to the SMT via universality. The IRCP is characterized by activated dynamical scaling: $L_\Omega \sim [\ln(\Omega_0/\Omega)]^{1/\psi}$. Here, Ω is the characteristic energy of the order-parameter fluctuations on length scale L_Ω, Ω_0 is a high-energy reference scale, and $\psi = 1/2$ is known as the tunneling exponent. The exponential length-energy relation implies that the dynamical exponent z is formally infinite. Moreover, the magnitude of the order-parameter fluctuations μ also scales logarithmically with energy, $\mu_\Omega \sim [\ln(\Omega_0/\Omega)]^\phi$, where the cluster exponent $\phi = (1 + \sqrt{5})/2$ is the golden ratio. Approaching criticality, the correlation length diverges as $\xi \sim |\delta|^{-\nu}$ where $\nu = 2$ and δ measures the relative distance to the critical point.

In this Letter, we study experimentally important transport properties at the pair-breaking SMT of disordered nanowires. We report both analytical and numerical calculations of the zero-temperature finite-frequency Aslamazov-Larkin [23] fluctuation corrections to the conductivity $\sigma(\omega)$. At criticality, the real part of the conductivity diverges as $\sigma'(\omega) \sim [\ln(\omega_0/\omega)]^{1/\psi}$ with vanishing frequency ω (ω_0 is a reference frequency). Off criticality, it satisfies the unconventional activated scaling form

$$\sigma'(\delta, \omega) = \frac{4e^2}{h}\left(\ln\frac{\omega_0}{\omega}\right)^{1/\psi} \Phi_\sigma\left(\delta^{\nu\psi} \ln\frac{\omega_0}{\omega}\right), \qquad (1)$$

where $\Phi_\sigma(x)$ is a universal scaling function. In the remainder of this Letter, we sketch the derivation of these results and discuss their experimental implications.

We begin by introducing a one-dimensional continuum model of Cooper pairs in the presence of Ohmic dissipation and disorder at $T = 0$ [7,8,10,18]

$$\mathcal{S} = \int dx \int d\tau \Big[D(x)|\partial_x \Psi(x,\tau)|^2 + \alpha(x)|\Psi(x,\tau)|^2 + \frac{u}{2}|\Psi(x,\tau)|^4 \Big] + \int dx \int \frac{d\omega}{2\pi} \gamma(x)|\omega||\tilde{\Psi}(x,\omega)|^2. \quad (2)$$

Here, $\tilde{\Psi}(x,\omega)$ is the Fourier transform of $\Psi(x,\tau)$, a complex superconducting order parameter at position x and imaginary time τ. We have explicitly included the random spatial dependence of all coupling constants. Stability and causality constrain u, $\gamma(x) > 0$, and we may choose a gauge where $D(x) > 0$. The quantum phase transition is tuned via $\delta \equiv \alpha - \alpha_c$ [10].

To proceed, we use a lattice discretization of the continuum action (2) in the limit of a large number of order-parameter components. This limit has no impact on the character of the critical point [10,18] and leads to a quadratic action

$$\mathcal{S}_0 = \sum_{i,j} \int \frac{d\omega}{2\pi} \tilde{\Psi}_i^*(\omega)(M_{ij} + |\omega|\delta_{ij})\tilde{\Psi}_j(\omega), \quad (3)$$

where the coupling matrix $M_{ij} \equiv (D_i/\sqrt{\gamma_i\gamma_j})\Delta_{ij}^2 + (r_i/\gamma_i)\delta_{ij}$ (with Δ_{ij} the discrete nearest neighbor Laplacian) must be determined self-consistently by solving $r_i = \alpha_i + (u/\gamma_i)\langle|\Psi_i(\tau)|^2\rangle$ and $\tilde{\Psi}_i(\omega) \to \tilde{\Psi}_i(\omega)/\sqrt{\gamma_i}$ has been rescaled. Note that the full effects of disorder can be realized while fixing $\gamma_i = \gamma$ to be constant [24].

We are now in a position to directly evaluate the dynamical conductivity via the Kubo formula [25]

$$\sigma(\omega) = -\frac{i}{\hbar\omega}\Big[\sum_{i,j}\int d\tau \langle J_i(\tau)J_j(0)\rangle e^{i\omega\tau} - \mathcal{D}\Big]_{i\omega\to\omega+i\eta},$$

where the current is given by $J_j(\tau) = (2ie/\gamma\hbar)D_j[\Psi_j^*(\tau)\Psi_{j+1}(\tau) - \Psi_{j+1}^*(\tau)\Psi_j(\tau)]$ with diamagnetic contribution $\mathcal{D} = (8e^2/\gamma\hbar)\sum_i D_i\langle|\Psi_i(0)|^2\rangle$. The Kubo formula can be evaluated by employing the spectral decomposition of M in terms of its eigenvector V_{ij} and accompanying diagonal eigenvalue $E_{ij} = \epsilon_i\delta_{ij}$ matrices defined by $\sum_k M_{ik}V_{kj} = V_{ij}\epsilon_j$. The resulting real part of the conductivity reads

$$\sigma'(\omega) = \frac{8e^2}{h}\sum_{a,b}\sum_{i,j} D_iD_j(V_{i,a}V_{j,a}V_{i+1,b}V_{j+1,b} - V_{i,a}V_{j+1,a}V_{i+1,b}V_{j,b})\mathcal{K}_{a,b}(\omega), \quad (4)$$

where

$$\mathcal{K}_{a,b}(\omega) = \Big\{-2\epsilon_a(\epsilon_a^2 - \epsilon_b^2 + \omega^2)\arctan(\omega/\epsilon_a) - 2\epsilon_b(-\epsilon_a^2 + \epsilon_b^2 + \omega^2)\arctan(\omega/\epsilon_b) + \omega(\epsilon_a^2 + \epsilon_b^2 + \omega^2)\ln\Big[\frac{(\epsilon_a^2+\omega^2)(\epsilon_b^2+\omega^2)}{\epsilon_a^2\epsilon_b^2}\Big]\Big\} \times [\epsilon_a^4 + 2\epsilon_a^2(-\epsilon_b^2+\omega^2) + (\epsilon_b^2+\omega^2)^2]^{-1}. \quad (5)$$

The validity of the scaling form (1) can be tested via a full numerical evaluation of Eqs. (4) and (5). This is possible by exploiting an efficient algorithm for computing the self-consistent pairing eigenmodes of $\mathcal{S}_0$ for large system sizes [10]. We have evaluated the conductivity (4) for chains of various length with up to 128 sites averaged over 3000 disorder realizations. For clarity, we limit our analysis to the largest size, $L = 128$, as the extrapolation to the thermodynamic limit $L \to \infty$ is nontrivial due to the crossover between ξ and L for the range of δ considered here. The results are displayed in Fig. 1.

For probe frequencies ω much larger than the characteristic fluctuation energy scale ω_0 of the chain, we fully saturate all quantum dynamics and observe a trivial $\sigma'(\omega) \sim 4e^2/(h\omega^2)$ conductivity. On the other hand, for $\omega \ll \omega_0$, σ' appears to be suppressed by a δ-dependent exponent. As we approach criticality ($\delta \to 0$) the functional form of the average conductivity is not easily ascertained from Eq. (4) due to the softening of critical modes, but the apparent disappearance of this exponent is fully consistent with the scaling theory in Eq. (1).

The predictions of Eq. (1) can be more thoroughly confirmed by searching for consistent scaling of the data shown in Fig. 1 after dividing by $[\ln(\omega_0/\omega)]^{1/\psi}$ and replotting as a function of the dimensionless scaling variable $x \equiv \delta^{\nu\psi}\ln(\omega_0/\omega)$. We find excellent data collapse over 5 orders of magnitude as shown in Fig. 2. From the scaled data, we extract the universal prefactor of the conductivity at the critical point, $\Phi_\sigma(0) = 0.70(4)$. Furthermore, an empirical analysis of the numerical scaling function in Fig. 2 suggests that $\Phi_\sigma(x \to \infty) \sim x^{-\nu}e^{-\mathcal{A}x}$ where $\mathcal{A} \sim O(1)$. The relevant limits of $\Phi(x)$ can also be inferred by appealing to the naive scaling prediction that in $d = 1$ the conductivity should be equal to $4e^2/h$ multiplied by a length. Activated dynamical scaling dictates that at criti-

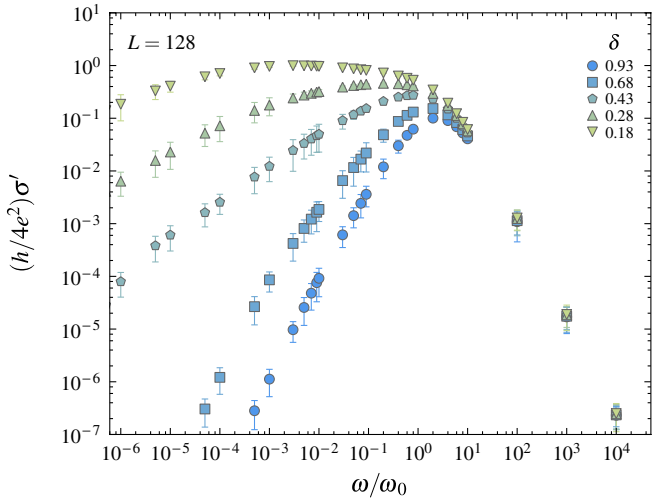

FIG. 1 (color online). The disorder averaged real conductivity for chains of 128 sites as a function of frequency measured in terms of a UV cutoff ω_0 for different values of the distance from the critical point, δ.

cality, lengths scale like $[\ln(\omega_0/\omega)]^{1/\psi}$ whereas in the Griffiths phase the relevant length scale is the correlation length $\xi \sim |\delta|^{-\nu}$, fixing the x-dependent power in front of the exponential in $\Phi_\sigma(x \to \infty)$ in order to cancel the logarithmic prefactor in Eq. (1).

Let us now turn to an analytical derivation of the dynamical conductivity. Near an IRCP, the conductivity will be dominated by large rare regions which are *locally* in the superconducting phase, i.e., by small clusters with exceptionally strong links D and typically small gaps r. In the low frequency limit, the effective links and gaps of these clusters can be quantified by a renormalization group analysis [18] that we invoke later. For now, we approximate each dominant cluster as a single two-site system with $r_{1,2}$ being the effective local gaps and D the effective link strength. In such a simple model, the conductivity Eq. (4) can be evaluated exactly, resulting in

$$\sigma'_{2\,\mathrm{site}} = \frac{8e^2D^2\ell^2}{\gamma^2\omega}\,\mathcal{K}_{12}(\omega), \tag{6}$$

where the eigenvalues of the 2×2 coupling matrix are $2\epsilon_{1,2} = [D/\gamma + (r_1 + r_2)/\gamma \pm \sqrt{(D/\gamma)^2 + (r_1 - r_2)^2/\gamma^2}]$ and we have introduced ℓ as the *length* of the link connecting the two sites. Because of the presence of the factor D^2 in Eq. (6), the average over the contributions of all two-site clusters will be dominated by those with an anomalously large links $D \gg r_{1,2}$. Hence, we may evaluate the conductivity by averaging

$$\sigma'(\Omega) = n_\Omega \int_0^\Omega dD \int_0^\infty d\ell P(D,\ell) \int_0^D dr R(r)\sigma'_{2\,\mathrm{site}}, \tag{7}$$

where $\Omega = \gamma\omega$ is the energy scale at which effective clusters with gap r connected by links of magnitude D and length ℓ appear with probability density $R(r)P(D,\ell)$, and n_Ω is the density of such clusters. The final step consists of using the asymptotic value of $\mathcal{K}_{12}(\omega) \sim \omega^{-1}\ln[\Omega/r]$, in the strong-disorder limit and $\Omega, D \gg r$, where $r = (r_1 + r_2)/2$. Using the values of $P(D,\ell)$, $R(r)$ and n_Ω at criticality [21,22]: $P(y) = \int d\ell P(y,\ell) = e^{-y}$, with $y = \ln(\Omega/D)/\ln(\Omega_0/\Omega)$, $R(y) = P(y)$ as well as the relation $n_\Omega^{-1} \sim \langle\ell\rangle \sim [\ln(\Omega_0/\Omega)]^{1/\psi}$ between density and average separation of clusters, we arrive at $\sigma' \sim n_\Omega\langle(D/\Omega)^2\ell^2\rangle\langle\ln(\Omega/r)\rangle \sim [\ln(\Omega_0/\Omega)]^{1/\psi}$, where $1/\psi = 2$ recovering Eq. (1) at criticality, where $\delta^{\nu\psi}\ln(\Omega_0/\Omega) \to 0$. Correlations between the links D and their lengths ℓ have no effect on the leading logarithmic divergence of this result at criticality. The analysis of Eq. (7) in the metallic and superconducting Griffiths phases crucially depends on the careful treatment of the correlations between ℓ and D. The resulting expressions are quite involved and will be discussed elsewhere [26].

We now discuss a subleading correction to the scaling of $\sigma'(\omega)$. In Eq. (7), the dissipative $z = 2$ dynamics causes the relation between energy Ω and the measured frequency ω to have a logarithmic correction [18] $\Omega = \gamma_0\mu_\Omega\omega$, where γ_0 is the bare dissipative coupling and μ_Ω is the mean value of superconducting order-parameter fluctuations. At criticality $\mu_\Omega \sim [\ln(\Omega_0/\Omega)]^\phi$ and $\ln(\Omega_0/\Omega) = \ln(\omega_0/\omega)$ up to log(log) corrections. In the metallic Griffiths phase, where presumably any real experiments on metallic nanowires would take place, $\mu_\Omega \sim \delta^{\nu\psi(1-\phi)}\ln(\Omega_0/\Omega)$ and the logarithms of energy and frequency are no longer simply equivalent. However, the exact value of μ_Ω can be obtained from the imaginary part of the dynamical order-parameter susceptibility [10,18], and its inclusion leads to quantitatively better data collapse as we extend the scaling theory of Eq. (1) deeper into the Griffiths regime [26].

In order to place the appearance of the link length ℓ in Eq. (6) and the average of Eq. (7) on firmer footing, we invoke the real space renormalization group technique of Refs. [18,27], providing direct access to the renormalization of the current operator. Starting with a chain described by the effective action of Eq. (3), we proceed by searching for the largest local coupling of the chain, $\Omega = \max\{r_i, D_i\}$. Suppose (i) $\Omega = r_2$, site 2 is then strongly fluctuating and can be integrated out of the system leading to an effective coupling $\tilde{D} = D_1D_2/r_2$ between sites 1 and 3. Their relative distance is given by $\tilde{\ell} = \ell_1 + \ell_2$, and the total current through clusters 1–3 is $\ell_1J_1 + \ell_2J_2 = \tilde{\ell}\tilde{J}$, where $\tilde{J} = (2ie/\gamma\hbar)\tilde{D}[\Psi_1^*\Psi_3 - \Psi_3^*\Psi_1]$. If on the other hand (ii) $\Omega = D_2$, sites 2 and 3 are strongly coupled forming an effective cluster $\tilde{2}$ in which the effective gap is $\tilde{r}_2 = r_2r_3/D_2$. The total current is then $\ell_1J_1 + \ell_2J_2 + \ell_3J_3 = \tilde{\ell}_1\tilde{J}_1 + \tilde{\ell}_2\tilde{J}_2$, where $\tilde{\ell}_1 = \ell_1 + \frac{1}{2}\ell_2$, $\tilde{\ell}_2 = \frac{1}{2}\ell_2 + \ell_3$, $\tilde{J}_1 = (2ie/\gamma\hbar)D_1[\Psi_1^*\Psi_{\tilde{2}} - \Psi_{\tilde{2}}^*\Psi_1]$ and $\tilde{J}_2 = (2ie/\gamma\hbar)D_3[\Psi_{\tilde{2}}^*\Psi_4 - \Psi_4^*\Psi_{\tilde{2}}]$. After process (i) or (ii), the energy scale Ω is lowered and the disorder of the effective system is increased [21]. Iterating this procedure leads to the probability distribution of gaps and links connecting

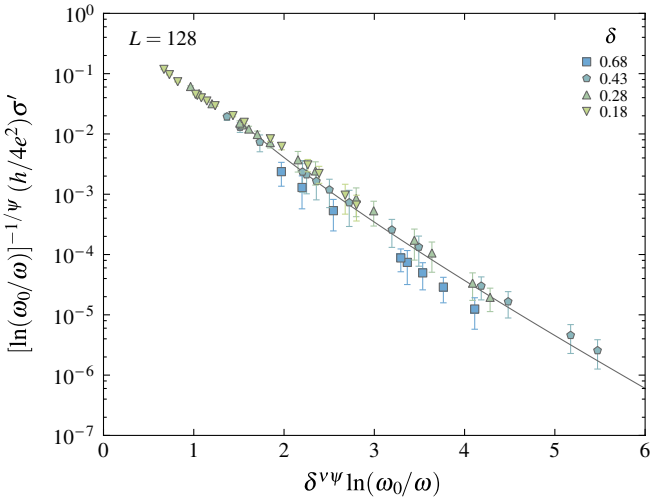

FIG. 2 (color online). The disorder averaged real conductivity scaling function Φ_σ as a function of the dimensionless scaling variable $x = \delta^{\nu\psi}\ln(\omega_0/\omega)$ for different values of δ as criticality is approached from above. The line is a guide to the eye showing the probable functional form of Φ_σ (see text).

the effective clusters as a function of energy [18] providing formal justification of Eq. (7).

We now compare our results with the transport properties of other systems governed by infinite randomness. To the best of our knowledge, only one other such system has been studied: spin conductivity in the dimerized antiferromagnetic spin-1/2 chain [27]. At criticality, the spin conductivity also diverges logarithmically, but with a weaker power than here, and is found to obey the scaling form $\sigma_{\rm spin}(\omega) \sim \ln(\omega_0/\omega)\Phi_{\rm spin}[\delta \ln(\omega_0/\omega)]$, where $\Phi_{\rm spin}(x \to 0) \approx 7/180$ and $\Phi_{\rm spin}(x \to \infty) \sim xe^{-2x}$. Although the thermodynamics of our system and the spin chain are funneled into the same universality class by the disorder, their transport properties are *not* universal because their underlying dynamics are different.

Finally, we highlight that the methods discussed here should also apply in higher dimensions. Specifically, at criticality, the system will also be governed by an infinite-randomness critical point [18,22] but with different exponents ψ, ν, and ϕ. The dynamical conductivity will likewise be dominated by rare and locally superconducting strongly coupled regions. Evaluating Eq. (7) immediately leads to $\sigma'(\omega) \sim [\ln(\omega_0/\omega)]^{(2-d)/\psi}$, since the spatial dimension enters explicitly only via the density of clusters $n_\Omega \sim L_\Omega^{-d}$, and the average conductivity of a two-site cluster $\langle \sigma'_{2\ \rm site} \rangle \sim [\ln(\omega_0/\omega)]^{2/\psi}$. As an immediate consequence, in the limit $\omega \to 0$, the critical conductivity vanishes for $d > 2$, and becomes constant at $d = 2$.

In conclusion, we have presented the numerically evaluated dynamical conductivity $\sigma'(\omega)$ for a model believed to describe the physics of disordered nanowires close to a superconductor-metal quantum phase transition and placed it in an analytical framework computed via the strong-disorder renormalization group. We have shown that $\sigma'(\omega)$ diverges logarithmically as $|\ln\omega|^2$ at criticality and obeys scaling in the metallic (Griffiths) phase with asymptotics dictated by naive dimensional analysis of physical quantities. Our results may be directly applicable to experimental transport measurements on thin nanowires which remain metallic as $T \to 0$, such as those reported in Ref. [6]. By studying wires of varying thickness at low temperatures, it may be possible to reach the critical regime where the logarithmic divergence of the fluctuation correction to the conductivity for $\omega > T$ could be directly observed. In $d = 2$, we expect σ' to be frequency-independent, consistent with experiments in disordered thin superconducting films (see, e.g., Ref. [28]). Finally, by presenting solid predictions arising from an effective action of strongly repulsive dissipative Cooperons, we lay open an avenue for the experimental investigation of the efficacy of such models when applied to dirty low-dimensional superconductors.

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[1] W. W. Webb and R. J. Warburton, Phys. Rev. Lett. **20**, 461 (1968).
[2] Y. Liu *et al.*, Science **294**, 2332 (2001).
[3] C. N. Lau *et al.*, Phys. Rev. Lett. **87**, 217003 (2001).
[4] G. R. Boogaard *et al.*, Phys. Rev. B **69**, 220503(R) (2004).
[5] A. Rogachev, A. T. Bollinger, and A. Bezryadin, Phys. Rev. Lett. **94**, 017004 (2005).
[6] F. Altomare *et al.*, Phys. Rev. Lett. **97**, 017001 (2006).
[7] S. Sachdev, P. Werner, and M. Troyer, Phys. Rev. Lett. **92**, 237003 (2004).
[8] A. Del Maestro, B. Rosenow, N. Shah, and S. Sachdev, Phys. Rev. B **77**, 180501(R) (2008).
[9] A. Del Maestro, B. Rosenow, and S. Sachdev, Ann. Phys. (N.Y.) **324**, 523 (2009).
[10] A. Del Maestro, B. Rosenow, M. Muller, and S. Sachdev, Phys. Rev. Lett. **101**, 035701 (2008).
[11] A. Rogachev *et al.*, Phys. Rev. Lett. **97**, 137001 (2006).
[12] M. V. Feigel'man and A. I. Larkin. Chem. Phys. **235**, 107 (1998).
[13] I. F. Herbut, Phys. Rev. Lett. **85**, 1532 (2000).
[14] A. V. Lopatin, N. Shah, and V. M. Vinokur, Phys. Rev. Lett. **94**, 037003 (2005); N. Shah and A. V. Lopatin, Phys. Rev. B **76**, 094511 (2007).
[15] B. Spivak, A. Zyuzin, and M. Hruska, Phys. Rev. B **64**, 132502 (2001).
[16] M. V. Feigel'man, A. I. Larkin, and M. A. Skvortsov, Phys. Rev. Lett. **86**, 1869 (2001); V. M. Galitski and A. I. Larkin, Phys. Rev. Lett. **87**, 087001 (2001).
[17] V. Galitski, Phys. Rev. Lett. **100**, 127001 (2008); Phys. Rev. B **77**, 100502(R) (2008).
[18] J. A. Hoyos, C. Kotabage, and T. Vojta, Phys. Rev. Lett. **99**, 230601 (2007); Phys. Rev. B **79**, 024401 (2009).
[19] T. Vojta and J. Schmalian, Phys. Rev. B **72**, 045438 (2005); T. Vojta, J. Phys. A **39**, R143 (2006).
[20] R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969); B. M. McCoy, Phys. Rev. Lett. **23**, 383 (1969).
[21] D. S. Fisher, Phys. Rev. Lett. **69**, 534 (1992); Phys. Rev. B **51**, 6411 (1995).
[22] F. Iglói, Phys. Rev. B **65**, 064416 (2002); F. Iglói and C. Monthus, Phys. Rep. **412**, 277 (2005).
[23] L. Aslamazov and A. Larkin, Sov. Phys. Solid State **10**, 875 (1968).
[24] J. R. Tucker and B. I. Halperin, Phys. Rev. B **3**, 3768 (1971).
[25] D. S. Fisher and P. A. Lee, Phys. Rev. B **23**, 6851 (1981).
[26] J. A. Hoyos *et al.* (unpublished).
[27] K. Damle, O. Motrunich, and D. A. Huse, Phys. Rev. Lett. **84**, 3434 (2000); Phys. Rev. B **63**, 134424 (2001).
[28] R. W. Crane *et al.*, Phys. Rev. B **75**, 094506 (2007).