Central Charges of Liouville and Toda Theories from $M5$-Branes

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We show that the central charge of the Liouville and Toda theories of type A, D, and E can be reproduced by equivariantly integrating the anomaly eight-form of the corresponding six-dimensional $\mathcal{N} = (0,2)$ theories, which describe the low-energy dynamics of $M5$-branes.

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Introduction.—$\mathcal{N} = 2$ supersymmetric field theories in four dimensions are very rich, from both the physical and mathematical points of view. Recently, it was observed in Ref. [1] that many $\mathcal{N} = 2$ theories can be understood in a unified manner by realizing them as a compactification of six-dimensional $\mathcal{N} = (0,2)$ theories on a Riemann surface. Furthermore, it was noted in Ref. [2] that Nekrasov's partition function [3] of such theories [with SU(2) gauge groups] computes the conformal blocks of the Virasoro algebra. It was also noted that the partition function on S^4, as given by Ref. [4], coincides with the corresponding correlation function of the Liouville theory. Soon this 2D–4D correspondence was extended in Refs. [5,6] to the case of SU(N) gauge groups where the Liouville theory generalizes to the A_{N-1} Toda theory [7].

Given that these 4D theories are engineered from theories on $M5$-branes, one would like to understand the above correspondence in terms of string or M theory. A step in this direction was made in Refs. [8,9]. Hinted at by the results of Refs. [5,10], in Ref. [9] an interesting observation was made, namely, that the anomaly eight-form of the 6D $\mathcal{N} = (0,2)$ theory of type A_{N-1} and the central charge of the Toda theory of the same type have similar structures:

$$I_8[A_{N-1}] = (N-1)I_8(1) + N(N^2-1)p_2(N)/24, \quad (1)$$

$$c_{\mathrm{Toda}}[A_{N-1}] = (N-1) + N(N^2-1)Q^2. \quad (2)$$

In this Letter, we show that (2) with the correct value for Q, namely, $Q = (\epsilon_1+\epsilon_2)^2/(\epsilon_1\epsilon_2)$, arises from (1) if we consider the compactification of the 6D (0, 2) theory on $\mathbb{R}^4$ with equivariant parameters $\epsilon_{1,2}$. Furthermore, we will see that this relation works for arbitrary theories of type A, D, and E.

Computation.—The anomaly eight-form of one $M5$-brane [11] is

$$I_8(1) = \tfrac{1}{48}\{p_2(NW) - p_2(TW) + \tfrac{1}{4}[p_1(TW) - p_1(NW)]^2\}, \quad (3)$$

where NW and TW stand for the normal and the tangent bundles of the worldvolume W, respectively, and p_k denotes the kth Pontryagin class. By using this, the anomaly of the $\mathcal{N} = (0,2)$ theory of type G ($G = A_n, D_n, E_n$) can be written as [12–15]

$$I_8[G] = r_G I_8(1) + d_G h_G \frac{p_2(NW)}{24}. \quad (4)$$

Here r_G, d_G, and h_G are the rank, the dimension, and the Coxeter number of the Lie algebra of type G, respectively. They are tabulated in Table I.

Now, we wrap the (0, 2) theory of type G on a four-manifold X_4. The 11D theory lives on:

$$\Sigma \times X_4 \times \mathbb{R}^5,$$

where Σ is the worldsheet of the resulting 2D theory. We take X_4 to be Euclidean and Σ to be Lorentzian. The supercharges decompose as

$$\mathbf{4}_+ \times \mathbf{4} \rightarrow (\tfrac{1}{2}, 2, 1, 2, \tfrac{1}{2}) + (\tfrac{1}{2}, 2, 1, 2, -\tfrac{1}{2}) + (-\tfrac{1}{2}, 1, 2, 2, \tfrac{1}{2}) + (-\tfrac{1}{2}, 1, 2, 2, -\tfrac{1}{2}),$$

where we listed the representation contents under the decomposition

$$\mathrm{SO}(5,1) \times \mathrm{SO}(5) \rightarrow \mathrm{SO}(1,1) \times \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \times \mathrm{SO}(3) \times \mathrm{SO}(2).$$

Here we have decomposed $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$ and $\mathrm{SO}(5) \supset \mathrm{SO}(3) \times \mathrm{SO}(2)$. The symplectic Majorana condition acts on each factor separately.

TABLE I. Data of the Lie algebras of type A, D, and E. Note that $r_G(h_G+1) = d_G$.

G	r_G	d_G	h_G
A_{N-1}	$N-1$	N^2-1	N
D_N	N	$N(2N-1)$	$2N-2$
E_6	6	78	12
E_7	7	133	18
E_8	8	248	30

Let us twist $\mathbb{R}^5$ over X_4 so that a fraction of the supersymmetry remains. We embed the spin connection of the $SU(2)_r$ factor into the SO(3) factor, that is,

$$SU(2)_r \longrightarrow \text{diagonal part of } [SU(2)_r \times SO(3)]. \quad (5)$$

Note that the SO(3) factor is the standard $SU(2)_R$ symmetry of the four-dimensional theory if we think of the setup as the compactification of the six-dimensional theory on Σ, giving an $\mathcal{N} = 2$ theory on X_4. Therefore this twist is the one used by Ref. [16].

After the twist, we get the symmetry group $SO(1,1) \times SU(2)_l \times SU(2)_r \times SO(2)$ and supercharges

$$(\tfrac{1}{2}, 2, 2, \tfrac{1}{2}) + (\tfrac{1}{2}, 2, 2, -\tfrac{1}{2}) + (-\tfrac{1}{2}, 1, 1+3, \tfrac{1}{2}) + (-\tfrac{1}{2}, 1, 1+3, -\tfrac{1}{2}).$$

The preserved supercharges [scalars under $SU(2)_l \times SU(2)_r$] form a two-dimensional $\mathcal{N} = (0,2)$ superalgebra, with U(1) R symmetry [17].

Let us exploit this 2D $\mathcal{N} = (0,2)$ superalgebra. We take the right movers to be the supersymmetric side. It is known that the anomaly polynomial and the central charges are related via

$$I_4 = \frac{c_R}{6} c_1(F)^2 + \frac{c_L - c_R}{24} p_1(T\Sigma), \quad (6)$$

where F is the external U(1) bundle which couples to the $U(1)_R$ symmetry. Let us check this formula against free multiplets. The anomaly polynomial of a right-moving complex Weyl fermion with charge q is

$$I_4 = \text{ch}(qF)\hat{A}(T\Sigma)|_4 = \frac{q^2}{2} c_1(F)^2 - \frac{p_1(T\Sigma)}{24}. \quad (7)$$

The right-moving chiral multiplet has one complex boson, whose anomaly is the same as that of two neutral Weyl fermions, and one Weyl fermion with charge 1. In total, $I_4 = c_1(F)^2/2 - p_1(T\Sigma)/8$ with $(c_L, c_R) = (0, 3)$. On the other hand, the left-moving free real boson has $I_4 = p_1(T\Sigma)/24$ with $(c_L, c_R) = (1, 0)$. Both cases agree with (6).

Now let us determine I_4 of the compactified theory by integrating I_8 over X_4. Let us assign the Chern roots as follows: $\pm t$ for the tangent bundle of Σ; $\pm\lambda_1$, $\pm\lambda_2$ for the tangent bundle of X_4; and $\pm n_1$, $\pm n_2$, 0 for the normal bundle. We include the U(1) R symmetry through

$$n_1 \longrightarrow 2c_1(F),$$

and the twisting (5) introduces

$$n_2 \longrightarrow \lambda_1 + \lambda_2. \quad (8)$$

Note that the doublet of $SU(2)_r$ has the Chern roots $\pm(\lambda_1 + \lambda_2)/2$. $(n_2, 0, -n_2)$ should then be the Chern roots of the triplet, resulting in (8).

Then we evaluate the anomaly polynomial. Notice that λ_1 and λ_2 will be integrated over X_4. Since the 2D spacetime effectively behaves as four-dimensional inside the anomaly polynomial, forms whose degree along $T\Sigma$ is higher than four automatically vanish. We get

$$I_4 = \left[\frac{r_G + 2d_G h_G}{12} \int (\lambda_1^2 + \lambda_2^2) + \frac{3r_G + 4d_G h_G}{12} \int \lambda_1 \lambda_2\right] c_1(F)^2 - \left[\frac{r_G}{48} \int (\lambda_1^2 + \lambda_2^2) + \frac{r_G}{48} \int \lambda_1 \lambda_2\right] p_1(T\Sigma).$$

Translating to $c_{L,R}$ using (6), we find

$$c_R = \tfrac{1}{2}[P_1(X_4) + 3\chi(X_4)]r_G + [P_1(X_4) + 2\chi(X_4)]d_G h_G,$$
$$c_L = \chi(X_4) r_G + [P_1(X_4) + 2\chi(X_4)]d_G h_G. \quad (9)$$

Here $\chi(X_4) = \int_{X_4} e(X_4)$ is the Euler number of X_4, and $P_1(X_4) = \int_{X_4} p_1(X_4)$ is the integrated first Pontryagin class which is 3 times the signature of X_4.

For example, let us wrap one $M5$-brane on $X_4 = K3$, in which case there is effectively no twisting. We start from $I_8(1)$ instead of $I_8[G]$, which effectively means using $r_G = 1$ and $d_G h_G = 0$ in (9). Using $P_1(K3) = -48$ and $\chi(K3) = 24$, we obtain

$$c_L = 24, \qquad c_R = 12,$$

which is the value for the heterotic string, as it should be.

The case we are most interested in is $X_4 = \mathbb{R}^4$, considering the characteristic classes in the equivariant sense [19]. We take the action of $U(1)^2$ to rotate two orthogonal two-planes in $\mathbb{R}^4$ and call the equivariant parameters $\epsilon_{1,2}$, respectively. The Chern classes of the two two-planes are $\epsilon_{1,2}$. Thus we have $p_1(T\mathbb{R}^4) = \epsilon_1^2 + \epsilon_2^2$ and $e(T\mathbb{R}^4) = \epsilon_1\epsilon_2$. We then use the localization formula, in the case where the fixed points are isolated:

$$\int_M \alpha = \sum_p \frac{\alpha|_p}{e(N_p)}.$$

The summation is over the fixed points p, and $e(N_p)$ is the equivariant Euler class of the normal bundle of p inside M. In our case the only fixed point is the origin. Therefore we have

$$P_1(\mathbb{R}^4) = \frac{\epsilon_1^2 + \epsilon_2^2}{\epsilon_1\epsilon_2}, \qquad \chi(\mathbb{R}^4) = 1. \quad (10)$$

Applying (9), we find

$$c_R = \frac{\epsilon_1^2 + 3\epsilon_1\epsilon_2 + \epsilon_2^2}{2\epsilon_1\epsilon_2} r_G + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} d_G h_G,$$
$$c_L = r_G + \frac{(\epsilon_1 + \epsilon_2)^2}{\epsilon_1\epsilon_2} d_G h_G. \quad (11)$$

Upon the identification $\epsilon_1/\epsilon_2 = b^2$ advocated in Ref. [2], c_L perfectly agrees with the central charge of the conformal Toda theory of type G [21]:

$$c_{\mathrm{Toda}}[G] = r_G + \left(b + \frac{1}{b}\right)^2 d_G h_G. \tag{12}$$

Discussion.—A couple of comments are in order. First, recall that in the construction of Ref. [1] the $\mathcal{N} = 2$ theories are obtained by wrapping $M5$-branes on $\mathbb{R}^4 \times \Sigma$, with a suitable twist on Σ which preserves one-half of the supersymmetry. So far, we have not taken this twist into account. When we perform it, the right-moving sector, which was the supersymmetric part, becomes topological and so $c_R \to 0$, while c_L is untouched and agrees with the central charge of the Liouville-Toda theories. This is consistent with the fact that Nekrasov's partition function computes the chiral half of the Liouville-Toda correlation functions.

Second, notice that Nekrasov's partition function was computed after introducing an equivariant deformation of $\mathbb{R}^4$ by a $\mathrm{U}(1)^2$ action with parameters $\epsilon_{1,2}$. More precisely, the symmetry of the 4D theory is

$$\mathrm{SO}(4) \times \mathrm{SU}(2)_R \simeq \mathrm{SU}(2)_l \times \mathrm{SU}(2)_r \times \mathrm{SU}(2)_R.$$

The topological theory has a modified Lorentz group

$$\mathrm{SO}(4)' \simeq \mathrm{SU}(2)_l \times \mathrm{SU}(2)_{r'},$$

where $\mathrm{SU}(2)_{r'}$ is the diagonal subgroup of $\mathrm{SU}(2)_r \times \mathrm{SU}(2)_R$. The $\mathrm{U}(1)^2$ used in the equivariant deformation is the Cartan subgroup of this modified $\mathrm{SO}(4)'$. This motivated our choice in (5). In view of this, it is also reasonable to evaluate the anomaly polynomial in the same equivariant sense [22]. It would be nice to have a better understanding of this point.

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Appendix: Central charges of Sicilian gauge theories of type A, D, and E.— In Ref. [10] the central charges a and c of the 4D superconformal Sicilian theories of A type (obtained by wrapping $M5$-branes on a genus-g Riemann surface), both in the $\mathcal{N} = 2$ and $\mathcal{N} = 1$ case, were computed from the 6D anomaly polynomial. We observe that from (4) the computation can be performed for the A, D, and E types.

Let us start with the $\mathcal{N} = 2$ case. By using the same Chern roots as before, the line bundle of the $\mathcal{N} = 1$ R symmetry is incorporated by $n_1 \to n_1 + \frac{2}{3} c_1(F)$, $n_2 \to n_2 + \frac{4}{3} c_1(F)$. $\mathcal{N} = 2$ supersymmetry requires $n_1 + t = 0$, $n_2 = 0$. The integral over the Riemann surface is $\int_\Sigma t = 2 - 2g$.

The 4D 't Hooft anomalies of $U(1)_R$ are read from the formula

$$I_6 = \frac{\mathrm{tr} R^3}{6} c_1(F)^3 - \frac{\mathrm{tr} R}{24} c_1(F) p_1(T_4). \tag{A1}$$

Comparing this with the integral of I_8, we get

$$\begin{aligned} \mathrm{tr}\, R^3 &= \tfrac{2}{27}(g-1)(13 r_G + 16 d_G h_G), \\ \mathrm{tr} R &= \tfrac{2}{3}(g-1) r_G. \end{aligned} \tag{A2}$$

Using the standard relations between a, c and $\mathrm{tr}R$, $\mathrm{tr}R^3$, we get

$$\begin{aligned} a &= (g-1)\frac{5 r_G + 8 d_G h_G}{24}, \\ c &= (g-1)\frac{r_G + 2 d_G h_G}{6}. \end{aligned} \tag{A3}$$

This agrees with Ref. [23] for the A series, and with Ref. [24] for the D series. Similar formulas can be obtained in the $\mathcal{N} = 1$ case. The R symmetry bundle is incorporated by $n_1 \to n_1 + c_1(F)$ and $n_2 \to n_2 + c_1(F)$, while $\mathcal{N} = 1$ supersymmetry requires $n_1 + n_2 + t = 0$. We get

$$\begin{aligned} a &= (g-1)\frac{6 r_G + 9 d_G h_G}{32}, \\ c &= (g-1)\frac{4 r_G + 9 d_G h_G}{32}. \end{aligned} \tag{A4}$$

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simplicity we consider only the Abelian case $U(1)^n$. Consider the space of differential forms on M valued in the polynomial of the formal parameters ϵ_a ($a = 1, \ldots, n$), and consider the deformed differential $D_\epsilon = d + \epsilon_a \iota_{k^a}$. Here ι is the interior product and k^a is the Killing vector of the ath U(1). Then ${D_\epsilon}^2 = \epsilon_a \mathcal{L}_{k_a}$, where $\mathcal{L}_{k_a}$ is the Lie derivative by k_a. We define the equivariant cohomology $H_{U(1)^n}(M)$ to be the cohomology of D_ϵ on the space of differential forms invariant under $U(1)^n$. Note that the formal parameters ϵ_a have degree 2. Equivariant characteristic classes are elements of the equivariant cohomology. For example, consider $\mathbb{C}$ acted on by U(1) which rotates the phase, and let the equivariant parameter be ϵ. The Chern class $c_1(T\mathbb{C})$ in the standard sense is of course trivial, but the equivariant Chern class is given by $c_1(T\mathbb{C}) = \epsilon$. For more details, see, e.g., [20].

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