How Small Can Thermal Machines Be? The Smallest Possible Refrigerator

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We investigate the fundamental dimensional limits to thermodynamic machines. In particular, we show that it is possible to construct self-contained refrigerators (i.e., not requiring external sources of work) consisting of only a small number of qubits and/or qutrits. We present three different models, consisting of two qubits, a qubit and a qutrit with nearest-neighbor interactions, and a single qutrit, respectively. We then investigate the fundamental limits to their performance; in particular, we show that it is possible to cool towards absolute zero.

When Carnot [1] set out to study the physics of steam engines—and in the process established thermodynamics—the key to progress was to abstract from real machines to idealized, "model independent" machines. He found that although the properties of each machine depend on the details of its construction, the fundamental limit to their efficiency is independent of such details. But can physics be left out completely? Here we return to physics and ask about other fundamental limits, specifically, is there a fundamental limit to their size? And, when they are small, are there additional constraints on their performance? Is there a complementarity between size and performance? For example, can small machines be constructed that cool arbitrarily close to absolute zero, or does size impose a fundamental limit?

In the present Letter we approach these questions in the framework of quantum mechanics which, importantly, provides a natural and universal notion of "size," namely, the dimension of the Hilbert space of the system. It is this measure of size that we will use here to characterize thermal machines. Obviously, spatial extent, mass, thermal capacity, etc., are all possible size measures, each with their own merits. Our measure has an informational flavor, motivated by the fundamental connection between information and thermodynamics [2,3].

The study of quantum heat engines is, of course, rather well developed [4–8]. In particular, direct quantum analogues of classical Carnot engines have been extensively studied [9,10], as well as Otto cycles [11–13]. However, these papers have in common the fact that they all use an external source of work and/or control—e.g., precise unitary transformations or macroscopic lasers [14]. In this Letter, however, we are interested in fundamental limits on the size of heat engines; hence, we must account for all degrees of freedom involved; we cannot allow for sources of external work or control. In other words, we want to study "self-contained" heat engines, focusing on refrigerators. Clearly no refrigerator can work without a supply of free energy; therefore, all we allow ourselves are two heat baths at differing temperatures $T_r < T_h$.

Apart from the implications for fundamental physics, this work is also relevant to other fields. In biology, for example, if cooling of the active site of a protein can be achieved then increased catalysis rates may be possible [15]. It is intriguing to ask whether simple mechanisms, such as those we describe here, are used by biological systems. A second field is nanotechnology, where the benefits from cooling at the atomic scale are clear.

Here we present three models: a refrigerator made of two qubits, one of a qubit and a qutrit with nearest neighbor interactions and one of a single qutrit—arguably the smallest possible. We focus on refrigeration of qubits but discuss more general objects also. Finally, we prove that there is no fundamental limit to how close towards absolute zero small refrigerators can cool.

Model I: Two qubits.—The first model consists of three qubits, two constitute the refrigerator, and one is the object to be cooled. It is inspired by algorithmic cooling [16], particularly by the few qubit version in [17].

Functioning principle.—Consider first two qubits, for simplicity taken initially to be immersed in the same bath at room temperature T_r, but later two different baths. Qubit 1 is the object to be cooled, while qubit 2 will eventually play the role of the "spiral" that takes heat from qubit 1 and dissipates it into the environment.

The free Hamiltonian for the two qubits is $H_0 = E_1 \Pi^{(1)} + E_2 \Pi^{(2)}$, where $\Pi^{(i)} = |1\rangle_i \langle 1|$ is a projector for qubit i, $|0\rangle_i$ are the ground states at zero energy, and $|1\rangle_i$ are the excited states at energies E_1 and E_2 respectively, where we take $E_2 > E_1$.

At equilibrium each qubit is in a thermal state τ_i,

$$\tau_i = r_i e^{-E_i \Pi^{(i)}/kT_r}, \tag{1}$$

where $r_i = (1 + e^{-E_i/kT_r})^{-1}$ and k is Boltzmann's constant. Since the qubits do not interact the total thermal state is simply the direct product state $\rho_{12} = \tau_1 \otimes \tau_2$.

A convenient way to represent the thermal state (1) is in terms of the probabilities, r_i and $1 - r_i$, to find the qubit in

the eigenstates $|0\rangle$ and $|1\rangle$ respectively, $\tau_i = r_i|0\rangle\langle0| + (1-r_i)|1\rangle\langle1|$.

We will denote by T_i^S the steady-state temperature of each qubit when refrigeration is occurring. Cooling of qubit 1 means reaching a temperature $T_1^S < T_r$, corresponding to a larger ground state probability $r_1^S > r_1$.

The idea of algorithmic cooling is to increase r_1^S by transferring excitations to the second qubit. Specifically, the probabilities of the eigenstates $|10\rangle$ and $|01\rangle$ of ρ_{12} are $(1-r_1)r_2$ and $r_1(1-r_2)$. Since we take $E_2 > E_1$, it follows that $(1-r_1)r_2 > (1-r_2)r_1$; i.e., we are more likely to be in the state $|10\rangle$ where qubit 1 is excited than in $|01\rangle$ where it is in the ground state. Suppose now that we apply a unitary U, which swaps these two states,

$$|10\rangle \leftrightarrow |01\rangle \qquad (2)$$

while leaving the others unchanged. After the swap we increase the ground state probability of qubit 1, therefore cooling it, and decrease the ground state probability of qubit 2, thus heating it. Since the qubits are in contact with a thermal bath they will eventually return to the environmental temperature T_r if we do nothing else. However, if we keep repeating U, qubits 1 and 2 will reach steady-state temperatures $T_1^S < T_r$ and $T_2^S > T_r$.

The procedure above, however, requires external work to be performed, since the energy of $|01\rangle$ is larger than that of $|10\rangle$. This is provided by some external system that induces the unitary transformation U; for example, in an NMR experiment this could be done via a sequence of pulses of an external magnetic field.

The idea behind our model is to replace the external work by a different source of free energy: Free energy can be provided whenever a system has access to two thermal baths at different temperatures. In our model this is accomplished by adding a third qubit, in contact with a thermal bath at a hotter temperature $T_h > T_r$. This qubit plays the role of the "engine." Thus our fridge consists in qubits 2 and 3 (the spiral and engine, respectively); qubit 1 is the object to be cooled.

To enable transitions between different states without an input of external energy, we take the engine qubit to have the energy level spacing E_3, such that $E_3 = E_2 - E_1$. With this condition we now have two degenerate energy eigenstates $|010\rangle$ and $|101\rangle$ which can be swapped without requiring work. The interchange

$$|101\rangle \leftrightarrow |010\rangle \qquad (3)$$

accomplishes on qubits 1 and 2 the transformation (2).

For transitions between $|010\rangle$ and $|101\rangle$ to occur we introduce an interaction Hamiltonian

$$H_{\text{int}} = g(|010\rangle\langle101| + |101\rangle\langle010|), \qquad (4)$$

with g the interaction strength. Note that the interaction is now via a Hamiltonian and not a unitary transformation. The distinction is important since a unitary necessarily implies external control—a device to implement the

correct timing, as well as a means of protecting the system from other external interactions. By using a Hamiltonian no such external control is required.

Furthermore, this interaction is taken to be weak compared to the free Hamiltonian $E_i \gg g$. In this regime the interaction will not significantly alter the energy eigenvalues or eigenvectors of the system (which remain governed by H_0) and hence we can meaningfully talk about the temperature of the individual qubits, since each will remain in the standard thermal form (1), with E_1 and E_2 the same as in the absence of interaction.

To understand the model first note that the interaction Hamiltonian can swap without impediment the states $|010\rangle$ and $|101\rangle$, since they are degenerate in energy insofar as the free Hamiltonian is concerned. However, if all qubits were kept at the same temperature, the system would be at equilibrium since the probability of the flips (3) is equal. To drive the transitions in one direction and cool qubit 1, we place the third qubit in a hotter bath; the probability of $|101\rangle$ becomes larger than that of $|010\rangle$, and so we enhance the probability of the forward flip in (3) and diminish that of the backward flip. It is this biasing of the interaction which takes heat from qubit 1 into qubit 2, creating a refrigerator.

Master equation.—The simplest way to model each qubit being in contact with a thermal bath is to imagine that with probability density p_i per unit time each qubit is thermalized back to its initial thermal state (1). (Note that for qubit 3 the bath temperature is now T_h.) Mathematically, we model this by the nonunitary evolution $\rho \mapsto p_i\delta t\tau_i \text{Tr}_i\rho + (1 - p_i\delta t)\rho$ in time δt. Here p_i quantifies how well insulated each particle is relative to the bath.

All together this leads to the master equation

$$\frac{\partial\rho}{\partial t} = -i[H_0 + H_{\text{int}}, \rho] + \sum_{i=1}^{3} p_i(\tau_i\text{Tr}_i\rho - \rho) \qquad (5)$$

(see supplementary information [18]), where $H_0 = E_1\Pi^{(1)} + E_2\Pi^{(2)} + E_3\Pi^{(3)}$. Note that in general the addition of H_{int} in (5) requires a modification of the dissipative term if it is to remain consistent [19]. However, we are interested only in the limit of vanishing g and p such that $g/p = \text{const}$. In this limit corrections to (5), of order pg, vanish, and the master equation is consistent.

Steady-state solution.—We will be interested in solving for the stationary state ρ_S, which satisfies $0 = -i[H_0 + H_{\text{int}}, \rho^S] + \sum_{i=1}^{3} p_i(\tau_i\text{Tr}_i\rho^S - \rho^S)$. The solution can easily be found analytically, but has a complicated dependence on all the parameters; therefore, it is much more illuminating to present a numerical analysis.

Figure 1 shows the temperature difference between qubit 1 (the system to be cooled) and its bath, as a function of the temperature of the hot bath T_h. The solid curve shows that when $T_h > T_r$ we are able to achieve cooling, i.e., $T_1^S < T_r$, and when $T_h = T_r$, that is when we supply no free energy, the temperature of qubit 1 is unchanged. In the dashed and dotted curve we show furthermore that if the
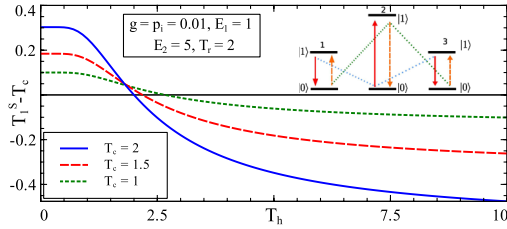
FIG. 1 (color online).   Cold qubit steady-state temperature difference $T_1^S - T_c$ versus hot bath temperature T_h, for various values of T_c. Inset: Schematic diagram of energy levels and interaction.

cold qubit is in contact with a colder bath (at temperature $T_c < T_r$), then we are still able to achieve cooling, i.e., $T_1^S < T_c$, provided T_h is hot enough. Therefore in all instances we demonstrate that our system acts as a refrigerator.

Until now we have focused solely on the task of cooling a qubit. One question is whether our fridge is able to cool more arbitrary objects than just a qubit. To achieve this, consider qubit 1 now as part of the fridge and use it to cool other objects. To see that is a valid viewpoint, in Fig. 2 we display the stationary heat current Q_1^S flowing between the bath and qubit 1. The heat current is defined as $Q_1^S = \mathrm{Tr}(H_1 \mathcal{D}_1(\rho^S))$ [20], where $\mathcal{D}_1(\rho^S) = p_1(\tau_1 \otimes \mathrm{Tr}_1 \rho^S - \rho^S)$ is the dissipator for qubit 1. We see that a positive current flows whenever the cold qubit is cooled below its bath temperature (cf. Fig. 1). Thus viewing the environment as the arbitrary object, we see that we are able to extract heat from it and therefore cool it. This may also be seen as independent confirmation that our system works as a refrigerator.

Parameter dependence and Zeno effects.—A natural question to ask is how the behavior of the fridge changes as we vary the parameters p_i independently. Indeed, in a "standard" refrigerator we do not want all parts to interact with the environment equally: the inside of the fridge has to be well insulated to maintain a low temperature while the spiral at the back of the fridge has to interact strongly with the environment to dissipate heat quickly.

Hence, for qubit 2 (the spiral), we expect that as p_2 becomes larger the performance of the fridge should increase (T_1^S should decrease). Furthermore, qubit 3 plays the role of the engine of the refrigerator, which it achieves by pumping heat from the hot environment into the system. We thus expect that the best performance is achieved when
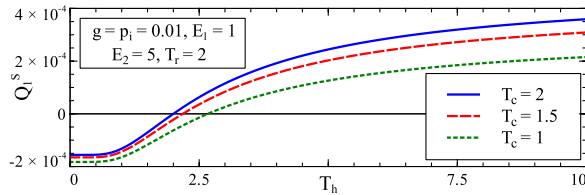
it interacts strongly with its environment, as this allows it to extract heat at the highest rate.

In Fig. 3 we plot the dependence of T_1^S on p_2 and p_3. For small values of p_2 and p_3 we indeed observe the expected behavior; however, as we increase them further the performance degrades. The reason is that the quantum Zeno effect [21] comes into play in the regime of strong coupling between the qubits and the environment. Thermalization is as though the environment measures each qubit. As we increase the rate of thermalization we enter a regime where the interaction H_{int} will not have time to work between successive thermalizations and hence the refrigerator is no longer able to function.

The dependence of T_1^S on p_1 is as expected; coldest temperatures are achieved in the limit of perfect insulation. See the supplementary information [18] for further details.

Approaching absolute zero.—An important question is whether or not there are fundamental limitations on the temperature to which we are able to cool the cold qubit. We show that no such limitations exist.

The minimal achievable temperature is limited by two effects: heat flowing into the fridge due to imperfect insulation, and the actual cooling ability (i.e., the ability to cool given perfect insulation). It is the second aspect which we are interested in.

We fix E_1, a characteristic of the object to be cooled and not of the refrigerator and also fix T_r, the environmental temperature. We increase E_3 (therefore E_2 also) and T_h such that the ratio E_3/T_h remains constant and much less than 1. This results in increasing the ground state probability of qubit 2, while maintaining a large excited state probability for qubit 3. Altogether this means that the interaction (4) becomes ever more biased as we increase E_3. This leads to cooling as close as we want towards absolute zero, as seen in Fig. 4.

Model II: One qubit, one qutrit.—One drawback of the previous model is that the interaction Hamiltonian (4) is a three-body interaction. Here we present a model with only two-body nearest neighbor interactions.

The model consists of three particles, where one is to be cooled and two construct the fridge. Particles 1 and 3 are qubits and particle 2 is a qutrit. The energy levels of each particle are such that the energy eigenstates $|020\rangle$ and $|101\rangle$ are degenerate. By introducing an interaction which can take the population of the latter into the former we can cool down qubit 1.
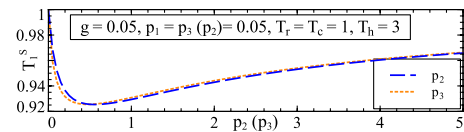

FIG. 2 (color online).   Stationary heat current Q_1^S of qubit 1 versus hot bath temperature T_h. Q_1^S becomes positive the moment the cold qubit reaches a temperature colder than its bath.

FIG. 3 (color online).   Qubit 1 steady-state temperature T_1^S against insulation parameter p_2 (p_3) in dashed (dotted) line. When p_2 (p_3) vanish we are unable to cool. For large p_2 (p_3) the performance of the fridge degrades due to a Zeno effect.
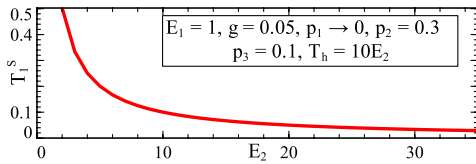
FIG. 4 (color online).   Qubit 1 steady-state temperature T_1^S against energy level spacing E_2. As the spacing increases the spin is cooled arbitrarily close to absolute zero.

We do this by introducing two separate interactions between the particles via the Hamiltonians

$$H_{\text{int}}^{(12)} = g(|02\rangle\langle 11| + |11\rangle\langle 02|) \otimes \mathbb{1}^{(3)}, \qquad (6)$$

$$H_{\text{int}}^{(23)} = h\mathbb{1}^{(1)} \otimes (|01\rangle\langle 10| + |10\rangle\langle 01|). \qquad (7)$$

Neither (6) nor (7) induces transitions between the two desired states. However, (6) causes transitions between $|020\rangle$ and $|110\rangle$ and (7) between $|110\rangle$ and $|101\rangle$. Therefore, in second order we induce the desired transition. Finally, we bias this interaction, as previously, by taking particles 1 and 2 to be in contact with a bath at temperature T_r and qubit 3 at T_h. This model behaves qualitatively the same as our previous one; details are given in the supplementary information [18].

The smallest possible fridge.—In the previous model the qutrit was taken to be in contact with a bath at temperature T_r. However, we could conceive of situations where each of its energy eigenstates has a different spatial distribution and so can be in contact with environments at differing temperatures. In such situations a smaller refrigerator can be constructed, by discarding the third qubit—the fridge now contains only a single qutrit, see Fig. 5. We believe this is the smallest possible system which may be called a refrigerator.

Conclusions.—We presented three simple models demonstrating that there is no fundamental difficulty in constructing small, self-contained refrigerators. Moreover, we showed that it is possible to cool towards absolute zero.

There are many interesting questions for the future. The first is what can be said about the efficiency of small refrigerators. Is our construction the most efficient, or are there other Hamiltonians which are better for cooling?

Moreover, it is fundamental to ask whether or not there exists a complementarity between small dimension and efficiency—can you only be large and efficient or small and inefficient? I.e., can a small machine reach the efficiency of an ideal Carnot engine? Our particular models do
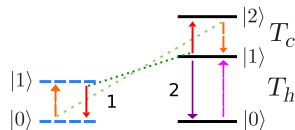
not reach this efficiency: both the spiral and the engine qubits reach stationary temperatures that differ by a finite (instead of infinitesimal) amount from the temperatures of their environments, which leads to irreversible heat exchanges. However, is there a better model? This is not clear, since in a Carnot cycle the system transitions through infinitely many states, not a finite number as we do here. The question is whether or not this affects the achievable efficiency of the refrigerator.

Finally, it would be interesting to study other thermal machines, for example, ones that produce "work."

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FIG. 5 (color online).   Schematic diagram of a fridge consisting of a single qutrit (particle 2) with the object to be cooled (particle 1)