Symmetries of Asymptotically Flat Four-Dimensional Spacetimes at Null Infinity Revisited

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It is shown that the symmetry algebra of asymptotically flat spacetimes at null infinity in 4 dimensions should be taken as the semidirect sum of supertranslations with infinitesimal local conformal transformations and not, as usually done, with the Lorentz algebra. As a consequence, two-dimensional conformal field theory techniques will play as fundamental a role in this context of direct physical interest as they do in three-dimensional anti–de Sitter gravity.

In the study of gravitational waves in the early 1960s [1,2], a lot of effort was devoted to specifying both local coordinate and global boundary conditions at null infinity that characterize asymptotically flat four-dimensional spacetimes. The group of nonsingular transformations leaving these conditions invariant is the well-known Bondi-Metzner-Sachs (BMS) group. It consists of the semidirect product of the group of globally defined conformal transformations of the unit 2-sphere, which is isomorphic to the orthochronous homogeneous Lorentz group, times the Abelian normal subgroup of so-called supertranslations.

What seems to have been largely overlooked so far is the fact that, when one focuses on infinitesimal transformations and does not require the associated finite transformations to be globally well-defined, the symmetry algebra is the semidirect sum of the infinitesimal local conformal transformations of the 2-sphere with the Abelian ideal of supertranslations, and now both factors are infinite-dimensional. This is already obvious from the details of the derivation of the asymptotic symmetry algebra by Sachs in 1962 [3].

Let $x^0 = u$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, and $A, B, \ldots = 2, 3$. Following Ref. [3] up to notation, the metric $g_{\mu\nu}$ of an asymptotically flat spacetime can be written in the form

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A - U^A du)$$
$$\times (dx^B - U^B du), \tag{1}$$

where β, V, U^A, and $g_{AB}(\det g_{AB})^{-1/2}$ are 6 functions of the coordinates, with $\det g_{AB} = r^4 b$ for a function $b(u, \theta, \phi)$. Sachs fixes $b = \sin^2\theta$, but the geometrical analysis by Penrose [4] suggests to keep it arbitrary throughout the analysis. In order to streamline the derivation below, it turns out to be convenient to use the parametrization $|b| = \frac{1}{4}e^{4\tilde\varphi}$, which implies, in particular, that $g^{AB}\partial_\alpha g_{AB} = \partial_\alpha \ln(\frac{r^4}{4}e^{4\tilde\varphi})$.

The falloff conditions for g_{AB} are

$$g_{AB}dx^A dx^B = r^2 \bar\gamma_{AB}dx^A dx^B + O(r), \tag{2}$$

where the two-dimensional metric $\bar\gamma_{AB}$ is conformal to the metric of the unit 2-sphere, $\bar\gamma_{AB} = e^{2\varphi}\,_0\gamma_{AB}$ and $_0\gamma_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2$. In terms of the standard complex coordinates $\zeta = e^{i\phi}\cot\frac{\theta}{2}$, the metric on the sphere is conformally flat: $d\theta^2 + \sin^2\theta d\phi^2 = P^{-2}d\zeta d\bar\zeta$, $P(\zeta, \bar\zeta) = \frac{1}{2}(1 + \zeta\bar\zeta)$. We thus have $\bar\gamma_{AB}dx^A dx^B = e^{2\tilde\varphi}d\zeta d\bar\zeta$, with $\tilde\varphi = \varphi - \ln P$. In the following, we denote by $\bar D_A$ the covariant derivative with respect to $\bar\gamma_{AB}$ and by $\bar\Delta$ the associated Laplacian.

In the general case, the remaining falloff conditions are

$$\beta = O(r^{-2}), \qquad U^A = O(r^{-2}),$$
$$V/r = -2rd_u\tilde\varphi + \bar\Delta\,\tilde\varphi + O(r^{-1}). \tag{3}$$

The transformations that leave the form of the metric (1) invariant up to a conformal rescaling of g_{AB}, i.e., up to a shift of $\tilde\varphi$ by $\tilde\omega(u, x^A)$, are generated by spacetime vectors satisfying

$$\mathcal{L}_\xi g_{rr} = 0, \qquad \mathcal{L}_\xi g_{rA} = 0, \qquad g^{AB}\mathcal{L}_\xi g_{AB} = 4\tilde\omega,$$
$$\mathcal{L}_\xi g_{ur} = O(r^{-2}), \qquad \mathcal{L}_\xi g_{uA} = O(1),$$
$$\mathcal{L}_\xi g_{AB} = O(r),$$
$$\mathcal{L}_\xi g_{uu} = -2r\partial_u\tilde\omega - 2\tilde\omega\,\bar\Delta\,\tilde\varphi + \bar\Delta\,\tilde\omega + O(r^{-1}). \tag{4}$$

The general solution to these equations is

$$\xi^u = f, \qquad \xi^A = Y^A + I^A,$$
$$I^A = -f_{,B}\int_r^\infty dr'(e^{2\beta}g^{AB}), \tag{5}$$
$$\xi^r = -\frac{1}{2}r(\bar D_A\xi^A - f_{,B}U^B + 2f\partial_u\tilde\varphi - 2\tilde\omega),$$

with $\partial_r f = 0 = \partial_r Y$. In addition,

$$\partial_u f = f\partial_u\tilde\varphi + \frac{1}{2}\psi - \tilde\omega$$
$$\Leftrightarrow f = e^{\tilde\varphi}\Big[T + \frac{1}{2}\int_0^u du' e^{-\tilde\varphi}(\psi - 2\tilde\omega)\Big], \tag{6}$$

where we use the notation $\psi = \bar{D}_A Y^A$ and where $\partial_u T = 0 = \partial_u Y^A$. Finally, Y^A is required to be a conformal Killing vector of $\bar{\gamma}_{AB}$.

The Lie algebra $\mathfrak{bms}_4$ can be defined as the semidirect sum of the Lie algebra of conformal Killing vectors $Y^A \frac{\partial}{\partial x^A}$ of the Riemann sphere with the Abelian ideal consisting of functions $T(x^A)$ on the Riemann sphere. The bracket is defined through $(\hat{Y}, \hat{T}) = [(Y_1, T_1), (Y_2, T_2)]$:

$$\begin{aligned}
\hat{Y}^A &= Y_1^B \partial_B Y_2^A - Y_1^B \partial_B Y_2^A, \\
\hat{T} &= Y_1^A \partial_A T_2 - Y_2^A \partial_A T_1 + \tfrac{1}{2}(T_1 \partial_A Y_2^A - T_2 \partial_A Y_1^A).
\end{aligned} \tag{7}$$

Consider then the modified Lie bracket

$$[\xi_1, \xi_2]_M = [\xi_1, \xi_2] - \delta^g_{\xi_1}\xi_2 + \delta^g_{\xi_2}\xi_1, \tag{8}$$

where $\delta^g_{\xi_1}\xi_2$ denotes the variation in ξ_2 under the variation of the metric induced by ξ_1, $\delta^g_{\xi_1} g_{\mu\nu} = \mathcal{L}_{\xi_1} g_{\mu\nu}$.

Let $\mathcal{I}$ be the real line times the Riemann sphere with coordinates $u, x^A = (\zeta, \bar{\zeta})$. On $\mathcal{I}$, consider the scalar field $\tilde{\varphi}, \tilde{\omega}$ and the vectors fields $\bar{\xi}(\tilde{\varphi}, \tilde{\omega}, T, Y) = f\frac{\partial}{\partial u} + Y^A \frac{\partial}{\partial x^A}$, with f given in (6) and Y^A an u-independent conformal Killing vector of the Riemann sphere.

When equipped with the modified bracket, both the vector fields $\bar{\xi}$ and the spacetime vectors (5) provide a linear representation of the direct sum of $\mathfrak{bms}_4$ with the Abelian algebra of conformal rescalings, i.e., the space of elements of the form (Y, T, ω), where $[(Y_1, T_1, \tilde{\omega}_1), (Y_2, T_2, \tilde{\omega}_2)] = (\hat{Y}, \hat{T}, \hat{\tilde{\omega}})$, with $\hat{Y}$ and $\hat{T}$ as before and $\hat{\tilde{\omega}} = 0$.

Depending on the space of functions under consideration, there are then basically two options which define what is actually meant by $\mathfrak{bms}_4$.

The first choice consists in restricting oneself to globally well-defined transformations on the unit or, equivalently, the Riemann sphere. This singles out the global conformal transformations, also called projective transformations, and the associated group is isomorphic to $SL(2, \mathbb{C})/\mathbb{Z}_2$, which is itself isomorphic to the proper, orthochronous Lorentz group. Associated with this choice, the functions $T(\theta, \phi)$, which are the generators of the so-called supertranslations, have been expanded into spherical harmonics. This choice has been adopted in the original work by Bondi, van der Burg, Metzner, and Sachs and followed ever since, most notably in the work of Penrose and Newman-Penrose [4,5]. A lot of attention has been devoted to the conformal rescalings and the "edth" operator together with spin-weighted spherical harmonics have been introduced. After attempts to cut this version of the BMS group down to the standard Poincaré group, it has been taken seriously as an invariance group of asymptotically flat spacetimes. Its consequences have been investigated, but we believe that it is fair to say that this version of the BMS group has had only a limited amount of success.

The second choice that we would like to advocate here is motivated by exactly the same considerations that are at the origin of the breakthrough in two-dimensional conformal quantum field theories [6]. It consists in focusing on local properties and allowing the set of all, not necessarily invertible, holomorphic mappings. In this case, Laurent series on the Riemann sphere are allowed. The general solution to the conformal Killing equations is $Y^\zeta = Y^\zeta(\zeta)$, $Y^{\bar{\zeta}} = Y^{\bar{\zeta}}(\bar{\zeta})$, and the standard basis vectors are chosen as

$$l_n = -\zeta^{n+1}\frac{\partial}{\partial \zeta}, \qquad \bar{l}_n = -\bar{\zeta}^{n+1}\frac{\partial}{\partial \bar{\zeta}}, \qquad n \in \mathbb{Z}. \tag{9}$$

At the same time, let us choose to expand the generators of the supertranslations with respect to

$$T_{m,n} = \zeta^m \bar{\zeta}^n, \qquad m, n \in \mathbb{Z}. \tag{10}$$

In terms of the basis vectors $l_l \equiv (l_l, 0)$ and $T_{mn} \equiv (0, T_{mn})$, the commutation relations for the complexified $\mathfrak{bms}_4$ algebra read

$$\begin{aligned}
&[l_m, l_n] = (m-n)l_{m+n}, \qquad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \\
&[l_m, \bar{l}_n] = 0, \qquad [l_l, T_{m,n}] = \left(\frac{l+1}{2} - m\right)T_{m+l,n}, \\
&[\bar{l}_l, T_{m,n}] = \left(\frac{l+1}{2} - n\right)T_{m,n+l}, \qquad [T_{l,m}, T_{n,o}] = 0.
\end{aligned} \tag{11}$$

The complexified Poincaré algebra is the subalgebra spanned by the generators

$$l_{-1}, l_0, l_1, \qquad \bar{l}_{-1}, \bar{l}_0, \bar{l}_1, \qquad T_{0,0}, T_{1,0}, T_{0,1}, T_{1,1}. \tag{12}$$

The considerations above apply for all choices of $\tilde{\varphi}$, which is freely at our disposal. In the original work of Bondi, van der Burg, Metzner, and Sachs, and in much of the subsequent work, the choice $\tilde{\varphi} = -\ln P$ was preferred. From the conformal point of view, the choice $\tilde{\varphi} = 0$ is interesting as it turns $\bar{\gamma}_{AB}$ into the standard flat metric on the Riemann sphere.

The consequences of local conformal invariance need to be taken into account when studying representations, and our result means that two-dimensional conformal field theory techniques should play a major role in both the classical and quantum theory of gravitational radiation. For instance, the representation theory based on the standard BMS group has been discussed in Ref. [7] and references therein, while related holographic considerations have appeared in Refs. [8,9]. Furthermore, implications of the supertranslations in the context of asymptotic quantization [10,11] have already been investigated. It should prove most interesting to extend these considerations to include the local conformal transformations.

A new perspective also arises for the problem of angular momentum in general relativity [12] since the factor algebra of $\mathfrak{bms}_4$ modulo the Abelian ideal of infinitesimal supertranslations is now the infinite-dimensional Virasoro algebra rather than the Lorentz algebra.

Earlier work where the relevance of conformal field theories for asymptotically flat spacetimes at null infinity has been discussed by starting out from the correspondence in the (anti–)de Sitter case includes Refs. [13–18]. In particular, a symmetry algebra of the kind that we have derived has been conjectured in Ref. [19].

A motivation for our investigation comes from Strominger's derivation [20] of the Bekenstein-Hawking entropy for black holes that have a near horizon geometry that is locally AdS_3. More recently, a similar analysis has been applied in the case of an extreme four-dimensional Kerr black hole [21]. Our hope is to make progress along these lines in the nonextreme case. As a first step, we have computed the behavior of Bondi's news tensor as well as the mass and angular momentum aspects under local conformal transformations in Ref. [22], where detailed proofs of all statements of this Letter can also be found. The next step consists in the construction of the surface charges, generators, and central extensions associated to $\mathfrak{bms}_4$.

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