Four-Qubit Entanglement Classification from String Theory

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We invoke the black-hole–qubit correspondence to derive the classification of four-qubit entanglement. The U-duality orbits resulting from timelike reduction of string theory from $D = 4$ to $D = 3$ yield 31 entanglement families, which reduce to nine up to permutation of the four qubits.

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Recent work has established some intriguing correspondences between two very different areas of theoretical physics: the entanglement of qubits in quantum information theory (QIT) and black holes in string theory. See [1] for a review. In particular, there is a one-to-one correspondence between the classification of three-qubit entanglement [2] and the classification of extremal black holes in the STU supergravity theory [3,4] that appears in the compactification of string theory from $D = 10$ to $D = 4$ dimensions. Moreover, the Bekenstein-Hawking black hole entropy is provided by the three-way entanglement measure.

The purpose of this Letter is to use this black-hole–qubit correspondence to address the much more difficult problem of classifying four-qubit entanglement, currently an active area of research in QIT as experimentalists now control entanglement with four qubits [5]. Although two and three-qubit entanglement is well understood, the literature on four qubits can be confusing and seemingly contradictory, as illustrated in Table I. This is due in part to genuine calculational disagreements, but in part to the use of distinct (but in principle consistent and complementary) perspectives on the criteria for classification. On the one hand there is the "covariant" approach which distinguishes the orbits of the equivalence group of stochastic local operations and classical communication (SLOCC) by the vanishing or not of covariants or invariants. This philosophy is adopted for the three-qubit case in [2,13], for example, where it was shown that three qubits can be tripartite entangled in two inequivalent ways, denoted W and GHZ (Greenberger-Horne-Zeilinger). The analogous four-qubit case was treated, with partial results, in [14]. On the other hand, there is the "normal form" approach which considers "families" of orbits. Any given state may be transformed into a unique normal form. If the normal form depends on some of the algebraically independent SLOCC invariants it constitutes a family of orbits parametrized by these invariants. On the other hand a parameter-independent family contains a single orbit. This philosophy is adopted for the four-qubit case

$$|\Psi\rangle = a_{ABCD}|ABCD\rangle \qquad A, B, C, D = 0, 1$$

in [11,12]. Up to permutation of the four qubits, these authors found six parameter-dependent families called G_{abcd}, L_{abc_2}, $L_{a_2b_2}$, $L_{a_3 0_{3\oplus\bar{1}}}$, L_{ab_3}, L_{a_4} and three parameter-independent families called $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$, $L_{0_{5\oplus\bar{3}}}$, $L_{0_{7\oplus\bar{1}}}$. For example, a family of orbits parametrized by all four of the algebraically independent SLOCC invariants is given by the normal form G_{abcd}:

$$\frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|1001\rangle + |0110\rangle), \tag{1}$$

where $a, b, c, d \in \mathbb{C}$. To illustrate the difference between these two approaches, consider the separable EPR-EPR state $(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$. Since this is obtained by setting $b = c = d = 0$ in (1) it belongs to the G_{abcd} family, whereas in the covariant approach it forms its own class. Similarly, a totally separable A-B-C-D state, such as $|0000\rangle$, for which all covariants/invariants vanish, belongs to the family L_{abc_2}, which also contains genuine four-way entangled states. These interpretational differences were also noted in [7].

Our string-theoretic framework lends itself naturally to the "normal form" perspective. We consider $D = 4$ supergravity theories in which the moduli parametrize a symmetric space of the form $M_4 = G_4/H_4$, where G_4 is the global U-duality group and H_4 is its maximal compact subgroup. After a further timelike reduction to $D = 3$ the moduli space becomes a pseudo-Riemannian symmetric space $M_3^* = G_3/H_3^*$, where G_3 is the $D = 3$ duality group and H_3^* is a noncompact real form of the maximal compact subgroup H_3. One finds that geodesic motion on M_3^* corresponds to stationary solutions of the $D = 4$ theory [15–20]. These geodesics are parametrized by the Lie algebra valued matrix of Noether charges Q and the problem of classifying the spherically symmetric extremal (nonextremal) black hole solutions consists of classifying the nilpotent (semisimple) orbits of Q. (Nilpotent means $Q^n = 0$ for some sufficiently large n.)

In the case of the STU model the $D = 3$ moduli space G_3/H_3^* is $SO(4,4)/[SL(2,\mathbb{R})]^4$ (a paraquaternionic manifold), which yields the Lie algebra decomposition

TABLE I. Various results on four-qubit entanglement.

Paradigm	Author	Year	Reference	Result modulo permutations		Result including permutations	
	Wallach	2004	[6]	?		90	
	Lamata *et al.*	2006	[7]	8 genuine	5 degenerate	16 genuine	18 degenerate
Classes	Cao *et al.*	2007	[8]	8 genuine	4 degenerate	8 genuine	15 degenerate
	Li *et al.*	2007	[9]	?		≥31 genuine	18 degenerate
	Akhtarshenas *et al.*	2010	[10]	?		11 genuine	6 degenerate
	Verstraete *et al.*	2002	[11]	9		?	
Families	Chterental *et al.*	2007	[12]	9		?	
	String theory	2010		9		31	

$$\mathfrak{so}(4,4) \cong [\mathfrak{sl}(2,\mathbb{R})]^4 \oplus (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}). \qquad (2)$$

The relevance of (2) to four qubits was pointed out in [1] and recently spelled out more clearly by Levay [20] who relates four qubits to $D = 4$ STU black holes. The Kostant-Sekiguchi correspondence [21] then implies that the nilpotent real orbits of SO(4, 4) acting on the adjoint representation **28** are in one-to-one correspondence with the nilpotent complex orbits of $[\mathrm{SL}(2,\mathbb{C})]^4$ acting on the fundamental representation (**2**, **2**, **2**, **2**) and hence with the classification of four-qubit entanglement. Note furthermore that it is the complex qubits that appear automatically, thereby relaxing the restriction to real qubits (sometimes called rebits) that featured in earlier versions of the black-hole–qubit correspondence.

Our main result, summarized in Table II, is that there are 31 entanglement families which reduce to nine up to permutations of the four qubits. From Table I we see that the nine agrees with [11,12] while the 31 is new. As far as we are aware, the nine four-qubit $[\mathrm{SL}(2,\mathbb{C})]^4$ cosets are also original.

The nilpotent orbits required by the Kostant-Sekiguchi theorem are those of $\mathrm{SO}_0(4,4)$, where the 0 subscript denotes the component connected to the identity. These orbits may be labeled by "signed" Young tableaux, often referred to as ab diagrams in the mathematics literature. See [22] and the references therein. Each signed Young tableau, as listed in Table II, actually corresponds to a single nilpotent O(4, 4) orbit of which the $\mathrm{SO}_0(4,4)$ nilpotent orbits are the connected components. Since O(4,4) has four components, for each nilpotent O(4,4) orbit there may be either 1, 2, or 4 nilpotent $\mathrm{SO}_0(4,4)$ orbits. This number is also determined by the corresponding signed Young tableau. If the middle sign of every odd length row is "−" ("+") there are 2 orbits and we label the diagram to its left (right) with a I or a II. If it only has even length rows there are 4 orbits and we label the diagram to both its left and right with a I or a II. If it is none of these it is said to be stable and there is only one orbit. The signed Young tableaux together with their labelings, as listed in Table II, give a total of 31 nilpotent $\mathrm{SO}_0(4,4)$ orbits, which are summarized in Fig. 1. We also supply the complete list of the associated cosets in Table II, some of which may be found in [18].

The STU model describes $\mathcal{N} = 2$ supergravity coupled to three vector multiplets and the Hawking temperature and Bekenstein-Hawking entropy of the STU black holes will depend on their mass and a maximum of eight charges (four electric and four magnetic). Through scalar dressing, these charges can be grouped into the $\mathcal{N} = 2$ central charge z and three "matter charges" z_a ($a = 1, 2, 3$), which exhibit a triality (corresponding to permutation of three of the qubits). The black holes are divided into extremal or nonextremal according as the temperature is zero or not. The orbits are nilpotent or semisimple, respectively. Depending on the values of the charges, the extremal black holes are further divided into small or large according as the entropy is zero or not. The small ones are termed lightlike, critical, or doubly critical according as the minimal number of representative electric or magnetic charges is 3, 2, or 1. The lightlike case is split into one 1/2-BPS solution, where the charges satisfy $z_1 = 0$, $|z|^2 = 4|z_2|^2 = 4|z_3|^2$, and three non-BPS solutions, where the central charges satisfy $z = 0$, $|z_1|^2 = 4|z_2|^2 = 4|z_3|^2$ or $z_2 = 0$, $|z_3|^2 = 4|z_1|^2 = 4|z|^2$ or $z_3 = 0$, $|z_2|^2 = 4|z_1|^2 = 4|z|^2$. The critical case splits into three 1/2-BPS solutions with $z = z_a \neq 0$, $z_b = z_c = 0$ and three non-BPS cases with $z = z_a \neq 0$, $z_b = z_c \neq 0$, where $a \neq b \neq c$. The doubly critical case is always 1/2-BPS with $|z|^2 = |z_1|^2 = |z_2|^2 = |z_3|^2$ and vanishing sum of the z_a phases. The large black holes may also be 1/2-BPS or non-BPS. One subtlety is that some extremal cases, termed "extremal," cannot be obtained as limits of nonextremal black holes. The matching of the extremal classes to the nilpotent orbits is given in Table II.

It follows from the Kostant-Sekiguchi theorem that there are 31 nilpotent orbits for the SLOCC-equivalence group acting on the representation space of four qubits. For each nilpotent orbit there is precisely one family of SLOCC orbits since each family contains one nilpotent orbit on setting all invariants to zero. The nilpotent orbits and their associated families are summarized in Table II, which is split into upper and lower sections according as the nilpotent orbits belong to parameter-dependent or parameter-independent families.

TABLE II. Each black hole nilpotent S0(4, 4) orbit corresponds to a 4-qubit nilpotent $[\mathrm{SL}(2,\mathbb{C})]^4$ orbit. z_H is the horizon value of the $\mathcal{N}=2$, $D=4$ central charge.

	STU black holes		$\dim_{\mathbb{R}}$	Four qubits		
Description	Young tableaux	$\mathrm{SO}_0(4,4)$ coset		$[\mathrm{SL}(2,\mathbb{C})]^4$ coset	Nilpotent representation	Family
Trivial	Trivial	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}_0(4,4)}$	1	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{[\mathrm{SL}(2,\mathbb{C})]^4}$	0	$\in G_{abcd}$
Doubly critical $\frac{1}{2}$ BPS	+− / −+ / + / − / + / −	$\frac{\mathrm{SO}_0(4,4)}{[\mathrm{SL}(2,\mathbb{R})\times \mathrm{SO}(2,2,\mathbb{R})]\ltimes[(\mathbf{2},\mathbf{4})^{(1)}\oplus\mathbf{1}^{(2)}]}$	10	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{[\mathrm{SO}(2,\mathbb{C})]^3\ltimes\mathbb{C}^4}$	$\|0110\rangle$	$\in L_{abc_2}$
	−+− / + / − / + / − / +	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(3,2;\mathbb{R})\ltimes[(\mathbf{5}\oplus\mathbf{1})^{(2)}]}$				
Critical, $\frac{1}{2}$ BPS and non-BPS	+−+ / − / + / − / + / −	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(2,3;\mathbb{R})\ltimes[(\mathbf{5}\oplus\mathbf{1})^{(2)}]}$	12	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{[\mathrm{SO}(3,\mathbb{C})\times\mathbb{C}]\times[\mathrm{SO}(2,\mathbb{C})\ltimes\mathbb{C}]}$	$\|0110\rangle + \|0011\rangle$	$\in L_{a_2b_2}$
	(I, II +− / −+ / +− / −+ I, II)	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{S}_P(4,\mathbb{R})\ltimes[(\mathbf{5}\oplus\mathbf{1})^{(2)}]}$				
Lightlike $\frac{1}{2}$ BPS and non-BPS	(I, II +−+ / −+ / +− / −) (−+− / +− / −+ / + I, II)	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SL}(2,\mathbb{R})\ltimes[(2\times\mathbf{2})^{(1)}\oplus(3\times\mathbf{1})^{(2)}\oplus\mathbf{2}^{(3)}]}$	16	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{[\mathrm{SO}(2,\mathbb{C})\ltimes\mathbb{C}]\times\mathbb{C}^2}$	$\|0110\rangle + \|0101\rangle$ $+\|0011\rangle$	$\in L_{a_2 0_{3\oplus\bar{1}}}$
Large non-BPS $z_H \neq 0$	−+− / +−+ / − / +	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(1,1,\mathbb{R})\times\mathrm{SO}(1,1,\mathbb{R})\ltimes[((\mathbf{2},\mathbf{2})\oplus(\mathbf{3},\mathbf{1}))^{(2)}\oplus\mathbf{1}^{(4)}]}$	18	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{\mathbb{C}^3}$	$\frac{i}{\sqrt{2}}(\|0001\rangle$ $+\|0010\rangle - \|0111\rangle$ $-\|1011\rangle)$	$\in L_{ab_3}$
	−+−+− / + / − / +	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(2,1;\mathbb{R})\ltimes[\mathbf{1}^{(2)}\oplus\mathbf{3}^{(4)}\oplus\mathbf{1}^{(6)}]}$				
''Extremal''	+−+−+ / − / + / −	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(1,2;\mathbb{R})\ltimes[\mathbf{1}^{(2)}\oplus\mathbf{3}^{(4)}\oplus\mathbf{1}^{(6)}]}$	20	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{\mathrm{SO}(2,\mathbb{C})\times\mathbb{C}}$	$i\|0001\rangle + \|0110\rangle$ $-i\|1011\rangle$	$\in L_{a_4}$
	(I, II +−+− / −+−+ I, II)	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{S}_p(2,\mathbb{R})\ltimes[\mathbf{1}^{(2)}\oplus\mathbf{3}^{(4)}\oplus\mathbf{1}^{(6)}]}$				
Large $\frac{1}{2}$ BPS and non-BPS $z_H = 0$	(I, II +−+ / +−+ / − / −) (−+− / −+− / + / + I, II)	$\frac{\mathrm{SO}_0(4,4)}{\mathrm{SO}(2,\mathbb{R})\times\mathrm{SO}(2,\mathbb{R})\ltimes[((\mathbf{2},\mathbf{2})\oplus(\mathbf{3},\mathbf{1}))^{(2)}\oplus\mathbf{1}^{(4)}]}$	18	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{[\mathrm{SO}(2,\mathbb{C})]^2\times\mathbb{C}}$	$\|0000\rangle + \|0111\rangle$	$\in L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$
"Extremal"	(I, II −+−+− / +−+) (+−+−+ / −+− I, II)	$\frac{\mathrm{SO}_0(4,4)}{\mathbb{R}^{3(2)}\oplus\mathbb{R}^{1(4)}\oplus\mathbb{R}^{2(6)}}$	22	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{\mathbb{C}}$	$\|0000\rangle + \|0101\rangle$ $+\|1000\rangle + \|1110\rangle$	$\in L_{0_{5\oplus 3}}$
"Extremal"	(I, II +−+−+−+ / −) (−+−+−+− / + I, II)	$\frac{\mathrm{SO}_0(4,4)}{\mathbb{R}^{(2)}\oplus\mathbb{R}^{2(6)}\oplus\mathbb{R}^{(10)}}$	24	$\frac{[\mathrm{SL}(2,\mathbb{C})]^4}{id}$	$\|0000\rangle + \|1011\rangle$ $+\|1101\rangle + \|1110\rangle$	$\in L_{0_{7\oplus\bar{1}}}$

If one allows for the permutation of the four qubits the connected components of each O(4, 4) orbit are reidentified reducing the count to 17. Moreover, these 17 are further grouped under this permutation symmetry into just nine nilpotent orbits. It is not difficult to show that these nine cosets match the nine families of [11,12], as listed in the final column of Table II (provided we adopt the version of L_{ab_3} presented in [12] rather than in [11]). For example, the state representative $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$,

$$|0111\rangle + |0000\rangle, \tag{3}$$

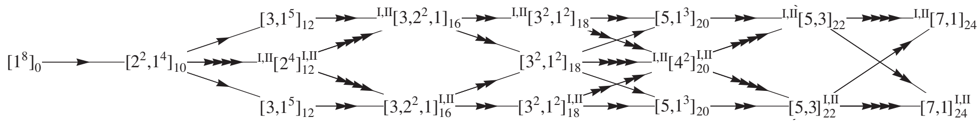

FIG. 1. SO(4, 4) Hasse diagram. The integers inside the bracket indicate the structure of the appropriate Young tableau. The subscript indicates the real dimension of the orbit. The arrows indicate their closure ordering defining a partial order [22].

is left invariant by the $[SO(2,\mathbb{C})]^2 \times \mathbb{C}$ subgroup, where $[SO(2,\mathbb{C})]^2$ is the stabilizer of the three-qubit GHZ state [13]. In contrast, the four-way entangled family $L_{0_{7\oplus\bar{1}}}$, which is the "principal" nilpotent orbit [21], is not left invariant by any subgroup. Note that the total of 31 does not follow trivially by permuting the qubits in these nine. Naive permutation produces far more than 31 candidates which then have to be reduced to SLOCC inequivalent families.

There is a satisfying consistency of this process with respect to the covariant approach. For example, the covariant classification has four biseparable classes A-GHZ, B-GHZ, C-GHZ, and D-GHZ which are then identified as a single class under the permutation symmetry. These four classes are in fact the four nilpotent orbits corresponding to the families $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$ in Table II, which are also identified as a single nilpotent orbit under permutations. Similarly, each of the four $A - W$ classes is a nilpotent orbit belonging to one of the four families labeled $L_{a_2 0_{3\oplus\bar{1}}}$ which are again identified under permutations. A less trivial example is given by the six A-B-EPR classes of the covariant classification. These all lie in the single family $L_{a_2 b_2}$ of [11], which is defined up to permutation. Consulting Table II we see that, when not allowing permutations, this family splits into six pieces, each containing one of the six A-B-EPR classes. Finally, the single totally separable class A-B-C-D is the single nilpotent orbit inside the single family L_{abc_2} which maps into itself under permutations.

Falsifiable predictions in the fields of high-energy physics or cosmology are hard to come by, especially for ambitious attempts, such as string or M theory, to accommodate all the fundamental interactions. In the field of quantum information theory, however, previous work has shown that the stringy black-hole–qubit correspondence can reproduce well-known results in the classification of two- and three-qubit entanglement. In this Letter this correspondence has been taken one step further to predict new results in the less well-understood case of four-qubit entanglement that can in principle be tested in the laboratory.

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