Example of a Quantum Anomaly in the Physics of Ultracold Gases

Maxim Olshanii,[1,2] Hélène Perrin,[2] and Vincent Lorent[2]

[1]*Department of Physics, University of Massachusetts Boston, Boston Massachusetts 02125, USA*
[2]*Laboratoire de physique des lasers, Institut Galilée, Université Paris 13 and CNRS,*
99 avenue J.-B. Clément, F-93430 Villetaneuse, France

In this Letter, we propose an experimental scheme for the observation of a quantum anomaly—-quantum-mechanical symmetry breaking—in a two-dimensional harmonically trapped Bose gas. The anomaly manifests itself in a shift of the monopole excitation frequency away from the value dictated by the Pitaevskii-Rosch dynamical symmetry [L. P. Pitaevskii and A. Rosch, Phys. Rev. A **55**, R853 (1997)]. While the corresponding classical Gross-Pitaevskii equation and the hydrodynamic equations derived from it do exhibit this symmetry, it is—as we show in our paper—violated under quantization. The resulting frequency shift is of the order of 1% of the carrier, well in reach for modern experimental techniques. We propose using the dipole oscillations as a frequency gauge.

PACS numbers: 67.85.−d, 03.65.Fd

Introduction.—The current decade is marked by the emerging links between ultracold-atom physics on one hand and cosmology and high-energy physics on another. Examples include kinetics of black holes [1], electron-positron pair production [2], *Zitterbewegung* [3], and the string theory limits posed on viscosity [4,5]. However, quantum anomalies are usually considered to be a purely quantum-field-theoretical phenomenon [6,7]. In this Letter, we suggest a scheme for observing a quantum anomaly in ultracold two-dimensional harmonically trapped Bose gas.

Quantum anomaly—otherwise known as quantum mechanical symmetry breaking—consists of three ingredients. The first ingredient is an exact symmetry in the classical version of the theory in question. The second ingredient is a divergence that appears in the straightforward quantum version of the theory. The third ingredient is a weak violation of the original symmetry that emerges in a regularized version of the quantum theory.

The two-dimensional δ potential has been long recognized as an example of quantum anomaly in elementary quantum mechanics [8,9]. The classical symmetry of the δ_2 potential originates from the following property: under the scaling transformation $r \rightarrow \lambda r$, the potential transforms in exactly the same way as the kinetic energy does. A consequence of this property is the absence of any length scale in the corresponding dynamical problem, both before and after a straightforward quantization. Next, an analysis of the scattering properties of the δ_2 potential [10,11] shows a divergence in an all Born orders of the scattering amplitude starting from the second. Finally, the subsequent regularization [10,11] leads to the appearance of the new length scale a_{2D}. The original symmetry becomes broken, and a quantum anomaly emerges.

In Ref. [12], Pitaevskii and Rosch predicted a dynamical symmetry that appears in the classical field theory (Gross-Pitaevskii equation) of the δ_2-interacting two-dimensional

harmonically trapped Bose gas. This symmetry is a direct consequence of the scaling symmetry of the δ_2 potential, described above. The consequences of this symmetry are (a) absence of the amplitude dependence of the main frequency (isochronicity) and (b) absence of higher overtones in the time dependence (monochromaticity) of the monopole oscillations of the moment of inertia. Both properties were demonstrated experimentally [13], along with an anomalously slow damping, for the case of a very elongated Bose-Einstein condensate; its Gross-Pitaevskii equation coincides with the one for a two-dimensional condensate.

In this Letter we address the question of whether the Pitaevskii-Rosch symmetry survives quantization.

In the fully quantized unitary gas, the analogous symmetry has been shown to remain unbroken [14]. The quantum correction to the frequency of the monopole excitation in an elongated condensate [13] was computed in Ref. [15].

The zero-temperature equation of state of the two-dimensional Bose gas.—The low-density zero-temperature quantum field theory (QFT) expression for the chemical potential of the two-dimensional Bose gas is well known [16,17]: it reads

$$\mu(n) \stackrel{\text{QFT}}{=} \frac{4\pi\hbar^2}{m} \chi(\pi e^{2\gamma+1} n a_{2D}^2) n, \tag{1}$$

where a_{2D} is the two-dimensional scattering length, $\gamma = 0.5772\ldots$ is the Euler's constant, n is the two-dimensional density, and $\chi(z) = \frac{1}{-W_{-1}(-z)}$. For a given two-dimensional two-body interaction potential $V(\mathbf{r})$, its scattering length a_{2D} is defined as the radius of a hard disk whose zero-energy s-wave scattering amplitude equals the one for the potential V. Here, $W_{-1}(z)$ is the minus-first branch of the Lambert's W function [18]. For the small values of the gas parameter na_{2D}^2, the factor χ is a logarithmically slow function of the density: $\chi(z) \stackrel{z \rightarrow 0}{\approx} 1/\ln(1/z) + \mathcal{O}[\ln(\ln(1/z))/\ln(1/z)^2]$. The expression (1) is an inverse

of a more conventional formula $n(\mu) = (m\mu/4\pi\hbar^2) \times \ln(4\hbar^2/e^{2\gamma+1}m\mu a_{2D}^2)$, shown to be the leading term in an expansion in powers of $1/\ln(4\hbar^2/e^{2\gamma+1}m\mu a_{2D}^2)$ [19].

In the case of three-dimensional short-range-interacting atoms tightly confined to a two-dimensional plane by a one-dimensional harmonic potential, the two-dimensional scattering length can be expressed through the three-dimensional scattering length a_{3D} and the confinement size $\tilde{a}_z = \sqrt{2\hbar/(m\omega_z)}$ as $a_{2D} = C_{2D}\tilde{a}_z \exp[-(\sqrt{\pi}/2) \times (\tilde{a}_z/a_{3D})]$, where $C_{2D} = 1.47\ldots$ [20]. The chemical potential becomes $\mu(n) \overset{\text{QFT}}{=} \bar{g}_{2D}n[\chi(\sigma(n)e^{-1/\epsilon})/\epsilon]$ where $\bar{g}_{2D} = 4\pi\epsilon\hbar^2/m$ is the "bare" two-dimensional coupling constant, $\epsilon = a_{3D}/\sqrt{\pi}\tilde{a}_z$ is the small parameter governing the proximity to the classical limit, and $\sigma(n) = \pi e^{2\gamma+1}(C_{2D})^2 n\tilde{a}_z^2$. In the limit $\epsilon \ll \min(1, 1/|\ln(\sigma(n))|)$, the factor χ approaches the density-independent constant ϵ, and the chemical potential converges to the prediction of the classical field theory (CFT), otherwise known as the Gross-Pitaevskii equation; there, the chemical potential reads

$$\mu(n) \overset{\text{CFT}}{=} \bar{g}_{2D}n. \tag{2}$$

Hydrodynamic equations.—The zero-temperature hydrodynamic (HD) equations for a two-dimensional harmonically trapped Bose gas read

$$\frac{\partial}{\partial t}n + \nabla_{\mathbf{r}}(n\boldsymbol{v}) = 0 \tag{3}$$

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla_{\mathbf{r}})\boldsymbol{v} = -(1/m)\nabla_{\mathbf{r}}[\mu(n) + V_{\text{HO}}(\mathbf{r})], \tag{4}$$

where m is the atomic mass, $n = n(\mathbf{r}, t)$ is the atomic density, $\boldsymbol{v} = \boldsymbol{v}(\mathbf{r}, t)$ is the atomic velocity, $V_{\text{HO}}(\mathbf{r}) = m\omega^2 r^2/2$ is the trapping potential energy per particle, $\mu(n)$ is the chemical potential, and $\mathbf{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$. Assuming that the gas is a Bose condensate with no vortices, the velocity field can be assumed to be irrotational: $\mathbf{v} = \hbar\nabla\Phi/m$, where $\Phi(\mathbf{r})$ is the potential of the velocity field. Under this assumption, the HD equations (3) and (4) can be rewritten in a Hamiltonian form, $\frac{\partial}{\partial t}n(\mathbf{r}, t) = \frac{i}{\hbar}[H, n(\mathbf{r}, t)]_{\text{HD}}$, $\frac{\partial}{\partial t}\Phi(\mathbf{r}, t) = \frac{i}{\hbar}[H, \Phi(\mathbf{r}, t)]_{\text{HD}}$. The Hamiltonian is represented by a sum of two parts, the first being in turn a sum of the kinetic and interaction energies and the second being the trapping energy: $H = H_0 + H_{\text{HO}}$. Here $H_0 = \int d^2\mathbf{r}\{\frac{\hbar^2}{2m}n(\nabla\Phi)^2 + \varepsilon(n)\}$, $H_{\text{HO}} = \int d^2\mathbf{r}V_{\text{HO}}(\mathbf{r})$, the "hydrodynamic commutator" $[\cdots]_{\text{HD}}$ is given by the Poisson brackets with respect to the (n, Φ) canonical pair,

$$
[A(n(\mathbf{r}, t), \Phi(\mathbf{r}, t)), B(n(\mathbf{r}', t), \Phi(\mathbf{r}', t))]_{\text{HD}} = -\frac{i}{\hbar}\left[\left(\frac{\partial}{\partial n}A(n(\mathbf{r}, t), \Phi(\mathbf{r}, t))\right)\left(\frac{\partial}{\partial \Phi}B(n(\mathbf{r}, t), \Phi(\mathbf{r}, t))\right)\right.
$$
$$
\left. - \left(\frac{\partial}{\partial \Phi}A(n(\mathbf{r}, t), \Phi(\mathbf{r}, t))\right)\left(\frac{\partial}{\partial n}B(n(\mathbf{r}, t), \Phi(\mathbf{r}, t))\right)\right]\delta_2(\mathbf{r} - \mathbf{r}'),
$$

and $\varepsilon(n) = \int_0^n dn'\mu(n')$ is the microscopic energy density.

The Pitaevskii-Rosch symmetry and the quantum anomaly at the HD level.—Introducing the generator of the scaling transformations, $Q = \frac{1}{\hbar}\int d^2\mathbf{r}n(\mathbf{r} \cdot \nabla\Phi)$ (see Ref. [12] for example), one obtains the following set of commutation relations:

$$[Q, H_0]_{\text{HD}} \overset{\text{QFT}}{=} 2iH_0 + ia_{2D}\frac{\partial}{\partial a_{2D}}H_0, \tag{5}$$

$[Q, H_{\text{HO}}]_{\text{HD}} \overset{\text{QFT}}{=} -2iH_{\text{HO}}$, $[H_{\text{HO}}, H_0]_{\text{HD}} \overset{\text{QFT}}{=} i\omega^2 Q$. Notice that the classical field theory chemical potential (2) does not depend on the two-dimensional scattering length a_{2D}. Then, the commutator (5) becomes

$$[Q, H_0]_{\text{HD}} \overset{\text{CFT}}{=} 2iH_0. \tag{6}$$

In this case, the observables H_0, H_{HO}, and Q form a closed three-dimensional algebra, identical to the one discovered by Pitaevskii and Rosch [12] at the Gross-Pitaevskii equation level.

However, the more accurate quantum field theory prediction for the chemical potential (1) depends explicitly on a_{2D}. According to the Eq. (5), the classical commutation relation (6) becomes corrected by $ia_{2D}\frac{\partial}{\partial a_{2D}}H_0$. Since the correction term is not generally expected to be a function

of three original members of the algebra, the algebra opens; such an opening constitutes a *quantum anomaly*.

Castin-Dum-Kagan-Surkov-Shlyapnikov equations.—If the factor χ in the chemical potential expression (1) was a constant, the HD equations (3) and (4) could be solved via the Castin-Dum-Kagan-Surkov-Shlyapnikov (CDKSS) scaling ansatz [21,22]. Note however, that χ is a very slow function of the density, and thus the scaling ansatz should approximately hold. Consider the following ansatz:

$$n(\mathbf{r}, t) = \frac{1}{\lambda(t)^2}n_0[1 - \{r/(\lambda(t)R)\}^2], \qquad \boldsymbol{v}(\mathbf{r}, t) = \boldsymbol{e}_r r\frac{\dot{\lambda}(t)}{\lambda(t)},$$

where $\tilde{R}_{\text{TF}} = \sqrt{2g_{2D}(n_0)n_0/m\omega^2}$ is the steady-state HD curvature of the density distribution [that also corresponds to the Thomas-Fermi radius of a gas with a density-independent coupling constant fixed to $g_{2D}(n_0)$], $g_{2D}(n) \equiv (\partial\mu/\partial n)$ is an effective density-dependent coupling constant, n_0 is the steady-state peak density; the scaling parameter λ obeys the CDKSS equation

$$\ddot{\lambda} = \frac{\omega^2 u(\lambda)}{\lambda^3} - \omega^2\lambda \tag{7}$$

with $u(\lambda) = g_{2D}(n_0/\lambda^2)/g_{2D}(n_0)$. It can be shown that the above ansatz solves the HD equations (3) and (4) almost everywhere, with the exception of an exponentially narrow ring close to the edge of the cloud.

Anomalous frequency shift of the monopole frequency.—
Linearization of the Eq. (7) for small excitation amplitudes
readily gives the frequency of small oscillations, $\Omega = \Omega_0[1 + \delta(\mathrm{AR}_\lambda = 0)]$, where [23]

$$\delta(\mathrm{AR}_\lambda = 0) = \frac{1}{4} \frac{\chi(\sigma(n_0)e^{-1/\epsilon})}{[1 - \chi(\sigma(n_0)e^{-1/\epsilon})]^2}. \qquad (8)$$

The deviation of the monopole frequency Ω from the clas-
sical field theory prediction $\Omega_0 \equiv \Omega|_{\epsilon \ll \min(1,1/|\ln(\sigma(n))|)} = 2\omega$ is a manifestation of the quantum anomaly.

Here and below, $\mathrm{AR}_\lambda = (\lambda_{\max} - \lambda_{\min})/(\lambda_{\max} + \lambda_{\min}) = (\lambda_{\max}^2 - 1)/(\lambda_{\max}^2 + 1)$ is the aspect ratio for the monopole
oscillations.

For $\epsilon \ll \min(1, \frac{1}{|\ln(\sigma(n_0))|})$, the relative anomalous cor-
rection converges to

$$\delta(\mathrm{AR}_\lambda = 0) \approx \frac{1}{4\sqrt{\pi}} \frac{a_{3D}}{\tilde{a}_z}. \qquad (9)$$

In particular, this estimate shows that the effect of the
anomaly can be enhanced using a Feshbach resonance.

Furthermore, a numerical analysis shows that under
quantization, the monopole frequency becomes amplitude
dependent. Figure 1 shows the corresponding prediction of
the CDKSS equation (7) compared to both the results of
the full HD treatment (3) and (4) and the limiting values (9).

*A possible experimental scheme for detecting the
anomalous frequency shift.*—Consider the following exci-
tation scheme. Initially, the cloud is prepared in the ground
state of a frequency $\omega_{\mathrm{init.}}$ trap at a position $x = 0, y = 0$. At
$t = 0$, the trap is simultaneously relaxed to a *lower* fre-
quency ω and shifted to a new position $x = x_0, y = 0$. This
initial condition will induce a superposition of a monopole
oscillation that starts from the lowest cloud size turning
point and a dipole oscillation—whose frequency ω does
not depend on either interaction strength or the oscillation
amplitude—that starts from the left (right) turning point for
$x_0 > 0(x_0 < 0)$.

What we suggest is to measure the spatial mean of
the square of the horizontal displacement with respect to
the new trap center, $s(t) = \int dx\,dy(x - x_0)^2 n(x, y, t)$. After
a lengthy but straightforward calculation it can be shown
that this observable will evolve in time as $s(t)/\tilde{R}_{\mathrm{TF}}^2 = P + A\cos[\Omega_0 t] + B\cos[\Omega_0\{1 + \delta(\mathrm{AR}_\lambda)\}t + \phi]$, where $P = (\lambda_{\max}^4 + 1)/12\lambda_{\max}^2 + \eta^2/2$, $A = \eta^2/2$, $B = (\lambda_{\max}^4 - 1)/12\lambda_{\max}^2$, $\phi = \pi$, $\eta = x_0/\tilde{R}_{\mathrm{TF}}$, and $\tilde{R}_{\mathrm{TF}}$ is the effective
Thomas-Fermi radius for the *final*, frequency ω trap (see a
definition above). Choosing $\eta = \sqrt{\frac{\lambda_{\max}^4 - 1}{6\lambda_{\max}^2}}$ leads to $A = B$. It
is easy to see that in this case, at $t = 0$ the beats between
the dipole and monopole oscillations have a node, with the
first crest reached in $1/(2\delta(\mathrm{AR}_\lambda))$ monopole periods. In
terms of the initial trap, the optimal shift of the trap can be
expressed as $x_0 = \frac{1}{\sqrt{6}}\sqrt{(\frac{\omega_{\mathrm{init.}}}{\omega})^2 - 1}\,\tilde{R}_{\mathrm{TF,init.}}$, where $\tilde{R}_{\mathrm{TF,init.}} = \sqrt{\omega/\omega_{\mathrm{init.}}}\,\tilde{R}_{\mathrm{TF}}$ is the Thomas-Fermi radius for the *initial*,
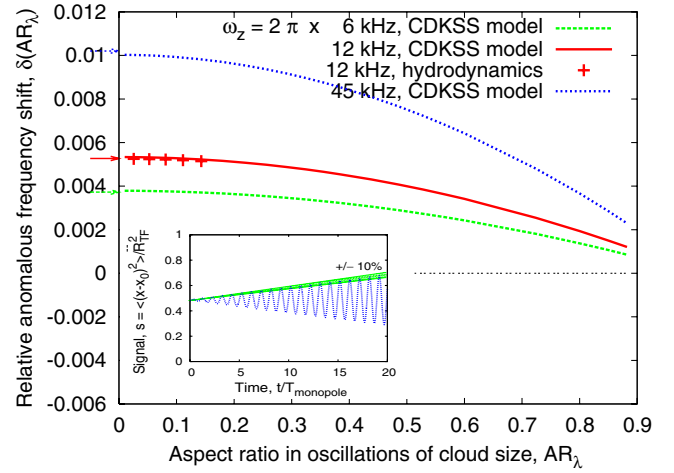frequency $\omega_{\mathrm{init.}}$ trap. Note as well that the right turning

FIG. 1 (color online). Relative anomalous shift of the mono-
pole frequency, $\delta(\mathrm{AR}_\lambda) \equiv (\Omega - \Omega_0)/\Omega_0$ as a function of the
amplitude of the excitation, for different values of the transverse
confinement frequency ω_z. Solid lines: prediction of the modi-
fied Castin-Dum-Kagan-Surkov-Shlyapnikov (CDKSS) model
(7). Crosses: prediction of the HD equations (3) and (4). Ar-
rows: the $a_{3D} \ll \tilde{a}_z$ limit (9) of the CDKSS model prediction (8)
for the zero-amplitude shift. The rest of the parameters corre-
sponds to $N = 10^3$ rubidium 87 atoms ($a_{3D} = 5.2$ nm) confined
longitudinally in a frequency ω harmonic trap, where $\omega = 2\pi \times 41$ Hz for the $\omega_z = 2\pi \times 45$ kHz data and $\omega = 2\pi \times 40$ Hz for
the rest of the data. The steady-state peak density n_0 entering the
CDKSS model was derived from the number of atoms N using
the Euler equation (4). A possible experimental setup corre-
sponding to the 45 kHz curve is described in detail in the
concluding section of this Letter. Insert shows the monopole
vs dipole beat signal for the optimal choice of the initial trap
shift (see main text). Parameters correspond to the 45 kHz curve
taken at the aspect ratio $\mathrm{AR}_\lambda = 0.486$. We also show the initial
linear part of the slow envelope, $(\lambda_{\max}^2/6 + [(\lambda_{\max}^4 - 1)/(12\lambda_{\max}^2)]\delta(\mathrm{AR}_\lambda = 0.486)\Omega_0 t$ plotted with a $\pm 10\%$ uncer-
tainty in the value of the anomalous correction $\delta(\mathrm{AR}_\lambda) = 0.00760$. The CDKSS model was used for calculations.

point for the oscillations of the scaling parameter can be
expressed as $\lambda_{\max} = \sqrt{\omega_{\mathrm{init.}}/\omega}$. In our excitation scheme,
the monopole oscillations start from the left turning point
that in turn corresponds to the lowest cloud size. The insert
of Fig. 1 presents an example of the projected beat signal
for a typical set of parameters.

Some residual anharmonicity and anisotropy may com-
plicate matters. For the anharmonicity, one can show that
for a quartic correction of a form $m\omega^2 r^2/2 \rightarrow (m\omega^2 r^2/2)(1 + \alpha r^2/2a_0^2)$ ($a_0 = \sqrt{\hbar/m\omega}$ being the size
of the ground state), the relative (to the monopole fre-
quency) shift of the monopole frequency is $\delta_{\mathrm{anharm.}} = \frac{12}{11}\alpha\sqrt{N\epsilon}$, to the leading order in both α and the amplitude.
The doubled dipole frequency—that serves as a refer-
ence—is unshifted in that order. On the other hand, the
anisotropy leads to a splitting of the dipole frequency with
the monopole frequency situated in between the two re-
sulting frequencies. Assume the doubled x-dipole fre-

quency $2\omega_x$ is chosen as a reference. Then the relative to this frequency monopole shift will be given, to the lowest order in both anisotropy $\omega_x - \omega_y$ and the amplitude, by $\delta_{\text{anisotr.}} = (\omega_x - \omega_y)/2\omega_{x,y}$, where $\omega_{x,y}$ is any of the two frequencies. Both the anharmonicity ($\delta_{\text{anharm.}}$) and anisotropy ($\delta_{\text{anisotr.}}$) shifts must be kept below 1% to allow for observation of the quantum anomaly.

Vortex-antivortex pair creation has been suggested as the dominant mechanism for damping of the two-dimensional monopole oscillations [2], all the conventional channels being suppressed due to the Pitaevskii-Rosch symmetry. Here, the amplitude of the oscillations decays as $\lambda_{\max}(t) = \lambda_{\max}(t)/(1 + t/\tau)^{1/10}$, where $\tau = \frac{737280}{127} (\frac{\omega_{\text{init.}}}{\omega})^5 (\frac{\lambda_{\min}(0)}{\lambda_{\max}(0)})^{10} \times \frac{\ln(4m\bar{g}_{\text{2D}}N/\sqrt{\pi}\hbar^2)}{(m\bar{g}_{\text{2D}}/\hbar^2)^{9/4}\sqrt{N}}$. Here, it is assumed that the amplitude of oscillations is large (i.e., $\lambda_{\max} \gg 1$) and the gas is dilute (i.e., $a_{\text{3D}} \ll \tilde{a}_z$).

Let us now propose a concrete example of a trap suited for an observation of the quantum anomaly. The numbers proposed are those of rubidium 87 in its $5S_{1/2}$, $F = 2$, $m_F = 2$ ground state. A blue detuned standing wave (wavelength 532 nm, power P, waist w_0) crosses, at a right angle, the symmetry axis z of a quadrupolar magnetic field; the magnetic field gradients are given by b' in the xy plane and $-2b'$ along z. This results in a series of pancake traps: there, the magnetic field is responsible for the weak horizontal trapping, while the standing wave gives a strong vertical confinement. Atoms could be loaded in one of the nodes of the standing wave, at a distance $z = d$ from the trap center; there the magnetic field is bounded from below by $B_0 = 2b'd$. By construction, this trap should be isotropic in the horizontal plane. Furthermore, the residual anisotropy defects could be further compensated by adding small magnetic gradients in the horizontal plane. For $b' = 167$ G/cm, $d = 400$ μm, $P = 4$ W, and $w_0 = 500$ μm, the oscillation frequencies are 41 Hz $\times$ 41 Hz $\times$ 45 kHz, and the magnetic field minimum $B_0 = 13.4$ G well prevents any Majorana losses. The photon scattering rate is as low as 2.6×10^{-3} s^{-1}, corresponding to 30 000 monopole mode oscillations. For $N = 10^4$ atoms (10 times the number used in Fig. 1), the anharmonic parameter is bounded from above by $\alpha = -2.1 \times 10^{-6}$. The predicted Thomas-Fermi radius of 15 μm is much smaller than both d and w_0; accordingly, the anharmonic shift has a negligibly small value of $\delta_{\text{anharm.}} = 5 \times 10^{-5}$. The chemical potential of $\mu/h = 1.7$ kHz is well below the transverse frequency; this ensures that the trap is sufficiently deep in the two-dimensional regime. Finally, taking the oscillation aspect ratio of $AR_\lambda = 0.486$ as an example, one obtains the damping time τ which is 1576 times longer than the period of the monopole oscillations. Overall, these figures bring our proposal within reach of modern experimental technology.

Summary and outlook.—In this Letter, we propose an experimental scheme for observation of a quantum anomaly in a two-dimensional harmonically trapped Bose gas.

The effect consists of a shift of the monopole excitation frequency away from the value dictated by the Pitaevskii-Rosch dynamical symmetry [12]. The shift we predict is only of the order of 1% of the base frequency. To detect it, we propose using the dipole oscillations as a reference frequency.

[1] L. J. Garay, J. R. Anglin, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **85**, 4643 (2000).
[2] P. O. Fedichev, U. R. Fischer, and A. Recati, *Phys. Rev. A* **68**, 011602 (2003).
[3] J. Y. Vaishnav and C. W. Clark, *Phys. Rev. Lett.* **100**, 153002 (2008).
[4] C. V. Johnson and P. Steinberg, *Phys. Today* **63**, No. 5, 29 (2010).
[5] J. E. Thomas, *Phys. Today* **63**, No. 5, 34 (2010).
[6] S. B. Treiman, R. Jackiw, B. Zumino, and E. Witten, *Current Algebras and Anomalies* (World Scientific, Singapore, 1985).
[7] J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of The Standard Model* (Cambridge University Press, Cambridge, England, 1992).
[8] R. Jackiw, in *M. A. B. Beg Memorial Volume*, edited by A. Ali and P. Hoodbhoy (World Scientific, New York, 1991).
[9] B. R. Holstein, *Am. J. Phys.* **61**, 142 (1993).
[10] L. R. Mead and J. Godines, *Am. J. Phys.* **59**, 935 (1991).
[11] S.-L. Nyeo, *Am. J. Phys.* **68**, 571 (2000).
[12] L. P. Pitaevskii and A. Rosch, *Phys. Rev. A* **55**, R853 (1997).
[13] F. Chevy, V. Bretin, P. Rosenbusch, K. W. Madison, and J. Dalibard, *Phys. Rev. Lett.* **88**, 250402 (2002).
[14] Y. Castin, *C.R. Physique* **5**, 407 (2004).
[15] L. Pitaevskii and S. Stringari, *Phys. Rev. Lett.* **81**, 4541 (1998).
[16] V. N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (Reidel, Dordrecht, 1983).
[17] C. Mora and Y. Castin, *Phys. Rev. A* **67**, 053615 (2003).
[18] $W_{-1}(z)$ is the real $W < -1/e$ solution of $z = We^W$ in the range $-1 < z < 0$.
[19] C. Mora and Y. Castin, *Phys. Rev. Lett.* **102**, 180404 (2009).
[20] D. S. Petrov and G. V. Shlyapnikov, *Phys. Rev. A* **64**, 012706 (2001).
[21] Y. Castin and R. Dum, *Phys. Rev. Lett.* **77**, 5315 (1996).
[22] Y. Kagan, E. L. Surkov, and G. V. Shlyapnikov, *Phys. Rev. A* **54**, R1753 (1996).
[23] Note that we have used the property $z\frac{d}{dz}\chi(z) = \chi(z)^2/(1 - \chi(z))$ that follows from the properties of the Lambert's W function.