Intermediate Disorder Regime for Directed Polymers in Dimension 1 + 1

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(Received 29 March 2010; published 24 August 2010)

We introduce a new disorder regime for directed polymers in dimension 1 + 1 by scaling the inverse temperature β with the length of the polymer n. We scale $\beta_n := \beta n^{-\alpha}$ for $\alpha \geq 0$. This scaling interpolates between the weak disorder ($\beta = 0$) and strong disorder regimes ($\beta > 0$). The fluctuation exponents ζ for the polymer end point and χ for the free energy depend on α in this regime, with $\alpha = 0$ corresponding to the Kardar-Parisi-Zhang polymer exponents $\zeta = 2/3$, $\chi = 1/3$, and $\alpha \geq 1/4$ corresponding to the simple random walk exponents $\zeta = 1/2$, $\chi = 0$. For $\alpha \in (0, 1/4)$ the exponents interpolate linearly between these two extremes. At $\alpha = 1/4$ we exactly identify the limiting distribution of the free energy and the end point of the polymer.

DOI: 10.1103/PhysRevLett.105.090603 PACS numbers: 05.10.Gg, 05.40.−a, 61.41.+e, 64.70.pj

Directed polymers in disordered media are a model of a great variety of physical phenomena ranging from vortex lines in superconductors [1], domain walls [2], roughness of crack interfaces [3], to Burgers turbulence [4] and the Kardar-Parisi-Zhang (KPZ) growth models [5]. Starting with [2,6], they have been the subject of intensive study [7,8] over the last 25 years. In the simplest setting, the statistics of directed polymers is described by a random probability distribution on the set of nearest neighbor random walks on the d-dimensional lattice. The randomness of the polymer statistics is related to a time-dependent random potential $V(i, s)$ which is given by an independent identically distributed (i.i.d.) collection of random variables placed on the sites of the $(d + 1)$-dimensional integer lattice. We assume that the V have mean zero and variance one. For a fixed realization of the disorder V (quenched disorder case), the energy of an n-step nearest neighbor walk S is

$$H_n(S) = \sum_{i=1}^{n} V(i, S_i).$$

Let $Z_n(\beta; x)$ be the point-to-point partition function

$$Z_n(\beta; x) = (2d)^{-n} \sum_{S:S_n=x} e^{-\beta H_n(S)},$$

and $Z_n(\beta)$ be the full partition function which is the sum of the point-to-point versions over x:

$$Z_n(\beta) = (2d)^{-n} \sum_{S} e^{-\beta H_n(S)} = \sum_{x} Z_n(\beta; x).$$

The distribution density for the polymer end point is thus

$$\rho_n^{\beta}(x) = Z_n(\beta; x)/Z_n(\beta).$$

The main goal is to study the asymptotic behavior of the free energy and the polymer end point as $n \to \infty$, and as β and d vary. At $\beta = 0$ the polymer is just the simple random walk; hence, it is entropy dominated and diffusive. For β large the polymer distribution localizes on paths of low energy and is no longer diffusive. A more precise separation between the two regimes is given in terms of the quenched and annealed free energies:

$$F_q(\beta) = \lim_{n\uparrow\infty} \frac{1}{n} \overline{\log Z_n(\beta)} \leq \lim_{n\uparrow\infty} \frac{1}{n} \log \overline{Z_n(\beta)} = \lambda(\beta).$$

Weak disorder corresponds to equality. The transition between weak and strong disorder occurs at a critical β_c. For $d = 1$ and 2, $\beta_c = 0$, while $0 < \beta_c \leq \infty$ for $d \geq 3$ [9]. The main contribution of this Letter is to show that in $d = 1$ one can observe a rich variety of different regimes interpolating between the weak and strong disorder by scaling the inverse temperature β with the length of the polymer n. We set $\beta_n := \beta n^{-\alpha}$, $0 \leq \alpha \leq 1/4$. Below we define the critical exponents and determine their values as α varies. It turns out that the asymptotic statistics are most interesting in the critical case $\alpha = 1/4$.

The behavior of the polymer is traditionally described in terms of fluctuation exponents for the polymer end point and the free energy: for the thermal average, $\langle S_n^2 \rangle \sim n^{2\zeta}$ and $\mathrm{Var}(\log Z_n(\beta)) \sim n^{2\chi}$. For (1 + 1)-dimensional polymers there are long-standing predictions: $\zeta = 2/3$ and $\chi = 1/3$ for all $\beta > 0$. These are the classical KPZ scalings [5,10]. Observe that $\zeta = 2/3$ and $\chi = 1/3$ satisfy the relation $\chi = 2\zeta - 1$, which is expected to hold whenever the polymers are localized. The fluctuations of the free energy are given by the asymptotic distributions for leading eigenvalues of random matrices. For example, in the point-to-point case, the fluctuations are given by the GUE Tracy-Widom distribution [11] F_{GUE}. More is true. After normalization, the collection of partition functions parametrized by the renormalized target point x converges to a universal stationary process called Airy_2 which has F_{GUE} as one-dimensional marginals; [12],

$$\log Z_n(\beta; n^{2/3}s) \sim c_1(\beta)n + c_2(\beta)n^{1/3}(\mathrm{Airy}_2(s) - a(\beta)s^2), \quad (1)$$

where $a(\beta)$ is a positive constant which compensates for a convex dependence of the free energy on the target point.

We also introduce the localization length exponent ν:

$$\langle S_n^2\rangle - \langle S_n\rangle^2 \sim n^{2\nu}.$$

In the strong disorder case $\nu = 0$ for typical environments; i.e., the polymer is localized. This does not contradict the fact that the variance averaged over environments is of order n [13]. For a few environments of total probability of order $n^{-1/3}$, $\langle S_n^2\rangle - \langle S_n\rangle^2$ is of order $n^{4/3}$. This explains the apparent discrepancy.

We now turn our attention to the "α-polymers" case with rescaled inverse temperature $\beta_n := \beta n^{-\alpha}$. For different values of $\alpha \geq 0$ they behave in ways that are quantifiably different from each other. For $\alpha > 1/4$ the polymer is diffusive and behaves as a simple random walk, with $\sqrt{n}$ fluctuations of the path, i.e., $\zeta = 1/2$, $\nu = 1/2$. The polymer, rescaled by $\sqrt{n}$, converges to Brownian motion with diffusivity constant 1 for almost all environments. As in the case $\beta = 0$, the exponent $\chi = 0$. For $0 \leq \alpha \leq 1/4$ the exponents ζ, χ, and ν depend on α as

$$\zeta(\alpha) = \tfrac{2}{3}(1-\alpha), \quad \chi(\alpha) = \tfrac{1}{3}(1-4\alpha), \quad \nu(\alpha) = 2\alpha, \quad (2)$$

linearly interpolating between the KPZ scaling at $\alpha = 0$ and the diffusive exponents $\zeta = 1/2$, $\chi = 0$, $\nu = 1/2$ at $\alpha = 1/4$. Note that the values of $\zeta(\alpha)$ and $\chi(\alpha)$ still satisfy $\chi(\alpha) = 2\zeta(\alpha) - 1$. The leading behavior of $\log Z_n(\beta n^{-\alpha})$ is of order $n^{1-2\alpha}$.

To derive these exponents we use the values of the critical exponents at $\alpha = 0$ together with the Airy process asymptotics of the point-to-point partition function. Let $E_n = \langle S_n\rangle$ and assume for the moment that $n^{2\nu}$ is of smaller order than E_n. This assumption means that most of the contribution to the fluctuations of the free energy comes from paths with an end point in a neighborhood of E_n. To leading order, the logarithm of the number of such paths is $-E_n^2/n$ and from (1) each path contributes energy on the order of $n^{1/3-\alpha}[\mathrm{Airy}_2(E_n/n^{2/3}) - \mathrm{Airy}_2(0)]$ to $\log Z_n(\beta; E_n)$, so that the log of the point-to-point partition function at E_n is

$$-\frac{E_n^2}{n} + n^{1/3-\alpha}[\mathrm{Airy}_2(E_n/n^{2/3}) - \mathrm{Airy}_2(0)]. \quad (3)$$

The polymer end point favors the value of E_n which maximizes (3). We obtain

$$E_n^2 \sim n^{4/3-\alpha}[\mathrm{Airy}_2(E_n/n^{2/3}) - \mathrm{Airy}_2(0)].$$

It follows that E_n is of a smaller order than $n^{2/3}$, and since on small distances the Airy_2 process is similar to the Wiener process we have $\mathrm{Airy}_2(E_n/n^{2/3}) - \mathrm{Airy}_2(0) \sim \sqrt{|E_n|/n^{2/3}}$. Hence

$$E_n^2 \sim n^{1-\alpha}\sqrt{|E_n|},$$

which gives $|E_n| \sim n^{(2/3)(1-\alpha)}$. Substituting this back into (3) implies that the fluctuations of the free energy are on the order of $n^{(1/3)(1-4\alpha)}$, which gives the formula for $\chi(\alpha)$. The localization length exponent is determined by the condition that the difference between logarithms of the point-to-point partition function with end points separated by the localization length n^ν should be of order 1. For distances $|x_1 - x_2| \sim n^{\nu(\alpha)}$ this difference is of order

$$n^{1/3-\alpha}[\mathrm{Airy}_2(x_1/n^{2/3}) - \mathrm{Airy}_2(x_2/n^{2/3})],$$

which is of order $\sqrt{x_1 - x_2}n^{-\alpha}$. This is of order 1 only if $\nu(\alpha) = 2\alpha$. We immediately see that for $\alpha < 1/4$ the polymer is localized ($\nu < \zeta$).

Using the Wiener process approximation the limiting distribution for the pair

$$\left(\frac{\log Z_n(\beta_n; xn^{\zeta(\alpha)}) - \log Z_n(\beta_n; 0)}{c_{\alpha,\beta}n^{\chi(\alpha)}}, \frac{S_n}{C_{\alpha,\beta}n^{\zeta(\alpha)}}\right) \quad (4)$$

is the joint distribution of the maximum value M of $W(t) - t^2$ and the point t_M where the maximum is achieved. Here $W(t)$ is a two-sided Brownian motion. This probability distribution was calculated exactly in [14]. The joint density of (t_M, M) at (t, a) is $g(|t|)h_a(|t|)\psi_a(0)$, where g has Fourier transform $\hat{g}(\lambda) = \int e^{i\lambda s}g(s)ds = \xi\mathrm{Ai}(i\xi\lambda)^{-1}$, h_a has Laplace transform

$$\hat{h}_a(\lambda) = \int_0^\infty e^{-\lambda s}h_a(s)ds = \mathrm{Ai}(\rho a + \xi)\mathrm{Ai}(\xi)^{-1},$$

and $\psi_a(x)$ has Fourier transform $\hat{g}(\lambda) = \int e^{i\lambda s}g(s)ds = \xi\mathrm{Ai}(i\xi\lambda)^{-1}$, and $\psi_a(x)$ has Fourier transform

$$\hat{\psi}_a(\lambda) = \pi\xi(\mathrm{Ai}(i\xi\lambda)\mathrm{Bi}(i\xi\lambda + \rho a) - \mathrm{Bi}(i\xi\lambda)\mathrm{Ai}(i\xi\lambda + \rho a))$$

with $\xi = 2^{-1/3}$, $\rho = 2^{2/3}$. Here Ai and Bi are the Airy functions [15].

We verify the formulas for $\zeta(\alpha)$ and $\chi(\alpha)$ by numerical simulations. See Fig. 1 for results.

Critical regime.—In the critical case $\alpha = 1/4$ the localization length is on the same order as the displacement of its end point ($\nu = \zeta$). The polymer is not localized anymore and the previous argument breaks down, although the exponents are still correct. The limiting distribution of the polymer end point is more involved but still computable. Notice that since $\chi(1/4) = 0$ we do not require any normalization. Consider the modified partition function

$$Z_n^*(\beta_n; x) = 2^{-n}\sum_{S:S_n=x}\prod_{i=1}^n(1 + \beta n^{-(1/4)}V(i, S_i)).$$

Expanding the product and summing over all n-step paths ending at x yields $Z_n^*(\beta_n; x) = 2^{-n}\sum_{k=0}^n \beta^k n^{-k/4}J_k^n(x)$, where

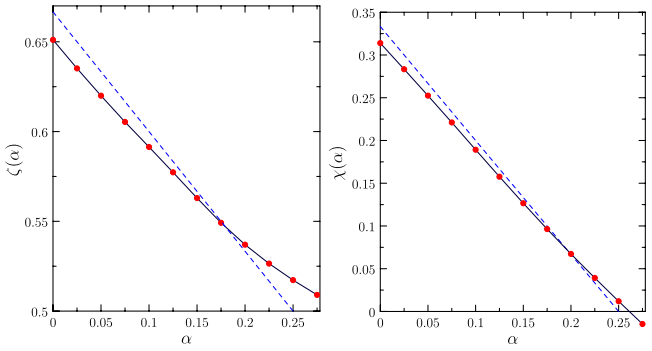

FIG. 1 (color online). Results of numerical computation of $\zeta(\alpha)$ and $\chi(\alpha)$ for 500 independent copies of the environment and polymer length $n = 5000$. We fixed $\beta = 1$ and let α vary from 0 to 0.275 in increments of 0.025. The environment distribution was chosen to be Gaussian. The dotted line corresponds to the predicted exponents (2).

$$J_k^n(x) = \sum p_x^n(i_k, x_k) \prod_{j=1}^{k} V(i_j, x_j) p(i_j - i_{j-1}, x_j - x_{j-1}).$$

Here $p(i, y) = \mathbf{P}(s(i) = y)$ for a simple random walk starting at zero, $p_x^n(i_k, x_k) = p(n - i_k, x - x_k)$, and the sum is over ordered $1 \le i_1 < \ldots < i_k \le n$ with $i_0 = 0$, and x_j, $1 \le j \le k$, with $x_0 = 0$. For fixed k,

$$2^{-n} n^{-k/4} J_k^n(x\sqrt{n})/p(n, x\sqrt{n}) \to 2^k I_k(x)/\varrho(1, x),$$

in distribution where $I_k(x)$ is given by

$$\int \varrho_x(t_k, x_k) \prod_{i=1}^{k} W(t_i, x_i) \varrho(t_i - t_{i-1}, x_i - x_{i-1}) dt_i dx_i. \tag{5}$$

Here $W(t, x)$ is a Gaussian white noise with $\overline{W(t,x)W(s,y)} = \delta(t-s)\delta(x-y)$, $\varrho(x, t) = \exp\{-x^2/2t\}/\sqrt{2\pi t}$, $\varrho_x(t_k, x_k) = \varrho(1 - t_k, x - x_k)$, and the integration is over $\{0 = t_0 < t_1 < \ldots t_k \le 1\} \times \mathbb{R}^k$ with $x_0 = 0$. The $n^{-k/4}$ term keeps the variance of order 1, and the 2^k terms come from the local central limit theorem for simple random walk. In the case $k = 1$ the convergence is easily seen by a Fourier transform computation. By the assumption that the random potential V has zero mean value and variance one, we have $\log \overline{e^{i\beta V}} = -\frac{1}{2}\beta^2(1 + o(1))$ as $\beta \to 0$. Hence

$$\log \overline{[\exp\{it n^{-(1/4)} J_1^n(x\sqrt{n})/p(n, x\sqrt{n})\}]}$$
$$= -\frac{t^2\beta^2}{2}(1 + o(1)) n^{-1/2} \sum_{i=1}^{n} \sum_{y} q_n^x(i, y)^2 \to -\frac{2t^2\beta^2}{\sqrt{\pi}}.$$

Here $q_n^x(i, y) = p(i, y)p(n - i, x\sqrt{n} - y)/p(n, x\sqrt{n})$. Therefore the limiting distribution is normal with mean zero and variance $4\beta^2/\sqrt{\pi}$, as is $2I_1(x)/\varrho(1, x)$. For $k > 1$, the convergence is handled by the general theory of U-statistics [16].

From these computations we conclude that $Z_n^*(\beta_n; x\sqrt{n})/p(n, x\sqrt{n})$ converges to the process

$$\mathcal{Z}(\beta, x) = \sum_{k \ge 0} (2\beta)^k I_k(x)/\varrho(1, x), \tag{6}$$

where $I_k(x)$ are given by (5). The same convergence also holds replacing Z^* by Z. Define the process $A_\beta(x) = \log \mathcal{Z}(\beta, x) - \log \varrho(1, x)$. This process is universal; it appears as a limit in the critical regime for polymer models with different distributions for the random potentials V. The statistics of the density function of the scaled polymer end point $S(n)/\sqrt{n}$ are the same as the statistics of the random density function

$$C_\beta \exp\{A_\beta(x) - x^2/2\}. \tag{7}$$

$A_\beta(x)$ interpolates between a Gaussian process ($\beta = 0$) and the Airy$_2$ process ($\beta = \infty$). To see this crossover one has to properly rescale the process with β to normalize the variance of its one-point distribution and its two-point correlation function at a fixed distance. The interpolation property follows from [17–19], where an exact formula for the one-point distribution of $A_\beta(x)$ was computed (see also [20]),

$$\mathbb{P}\{A_\beta(x) \ge s\} = \int e^{-e^{-r}} f(s + 2\beta/3 - \log\sqrt{32\pi\beta^4} - r) dr$$

where,

$$f(r) = \kappa^{-1} \det(I - K) \mathrm{tr}((I - K)^{-1} P_{\mathrm{Airy}})$$

with $\kappa = 2\beta^{4/3}$, $P_{\mathrm{Airy}}(x, y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, and

$$K(x, y) = \int (1 - e^{-\kappa t})^{-1} \mathrm{Ai}(x + t)\mathrm{Ai}(y + t) dt.$$

It follows that $\mathbb{P}\{A_\beta(x) \le 2\beta^{4/3} s\} \to F_{\mathrm{GUE}}(s)$. Finally, the process $A_\beta(x)$ is continuous and locally Brownian.

The limiting distribution is also characterized [17,18] by observing that $\mathcal{Z}(\beta, x)$ is the Wick exponential of a stochastic integral. Let B be a one-dimensional Brownian motion that is independent of the white noise $W(t, x)$. Let $\langle \cdot \rangle_{x,T}$ be the Wiener path integral over paths starting at zero and ending at x at time T. Define

$$\mathcal{Z}(T, \beta, x) = \left\langle : \exp : \left\{ 2\beta \int_0^T W(s, B(s)) ds \right\} \right\rangle_{x,T}. \tag{8}$$

By Brownian scaling, $\mathcal{Z}(T, \beta, x)$ is equal in distribution to $T^{-1/2}\mathcal{Z}(1, T^{1/4}\beta, T^{-1/2}x)$, and expanding the exponential we see that $\mathcal{Z}(\beta, x)$ from (6) is the same as $\mathcal{Z}(1, \beta, x)$ from (8). The continuum point-to-line partition function $\mathcal{Z}_\ell(T, \beta)$ is obtained by dropping the restriction that $B(T) = x$ in (8).

In the critical regime $\alpha = 1/4$, the discrete polymer scales to the continuum random polymer with weights with respect to the standard Wiener path integral given formally by the integrand in (8).

Writing $\langle\cdot\rangle_{T,\beta}$ for the new path integral, (8) provides another derivation of the fluctuation exponents $\zeta(\alpha)$ and $\chi(\alpha)$. Setting $\beta = T^{-\alpha}$ and making the change of variables $u = T^{-4\alpha}s$ gives, by standard Brownian scaling,

$$\mathcal{Z}(T, T^{-\alpha}, x) \equiv \mathcal{Z}(T^{1-4\alpha}, 1, xT^{-2\alpha}), \tag{9}$$

in the sense of equality in distribution, so that integrating out the end point X, $\mathcal{Z}_\ell(T, T^{-\alpha}) \equiv \mathcal{Z}_\ell(T^{1-4\alpha}, 1)$. The relation $\mathrm{Var}(\log \mathcal{Z}_\ell(T, \beta)) \sim T^{2/3}$ then shows that

$$\mathrm{Var}\left(\log \mathcal{Z}_\ell(T, T^{-\alpha})\right) \sim T^{(2/3)(1-4\alpha)},$$

which agrees with $\chi(\alpha)$. For $\zeta(\alpha)$ we realize

$$\langle B_T^2\rangle_{T,\beta} = \int_{-\infty}^{\infty} x^2 \mathcal{Z}(T, \beta, x)\rho(T, x)dx \sim T^{4/3}.$$

Setting $\beta = T^{-\alpha}$ we get from (9)

$$\langle B_T^2\rangle_{T,T^{-\alpha}} = \int_{-\infty}^{\infty} x^2 \mathcal{Z}(T, T^{-\alpha}, x)\rho(T, x)dx \sim T^{(4/3)(1-\alpha)}.$$

Note that $h_\beta(T, x) = -\log \mathcal{Z}(T, \beta, x)$ satisfies the KPZ equation

$$\partial_T h_\beta = -\tfrac{1}{2}(\partial_x h_\beta)^2 + \tfrac{1}{2}\partial_x^2 h_\beta + \beta W.$$

From (7), in the critical disorder regime, the polymer end point deposits itself on the real line according to the height of the KPZ interface.

In terms of physical applications, most, though not all, models have an adjustable parameter which can be scaled as we have done to obtain the intermediate disorder regimes. An example where this is not the case is the polynuclear growth model [21]. A positive example is the original motivation for the directed polymer model, domain walls in the two dimensional random field Ising model, which models pinning by impurities. One studies it at low temperature starting with a wall between + and −. The strength of the random field can be adjusted and corresponds to the rescaled temperature in our random polymer model. Domain walls are modeled by the polymer paths. In $d = 1 + 1$, the exponents $\zeta = 2/3$, $\xi = 1/3$ are well established numerically. When the adjustable parameter is present, one can obtain the intermediate disorder regimes as the appropriately chosen asymptotics. The continuum random polymer is a universal limit at the critical weak scaling. Therefore one can now understand fluctuations of these models in this asymptotic regime, and, in particular, see the explicit transition from Gaussian behavior, with diffusive exponents, to the non-Gaussian, coupled behavior, with nondiffusive exponents.

Summarizing, we have identified a new disorder regime for directed polymers in $(1 + 1)$ dimensions and computed the wandering exponent ζ and free energy fluctuation exponent χ, in addition to the localization length exponent ν. At $\alpha = 1/4$ the exponents are diffusive but the fluctuations are strongly coupled to the disorder variables V and are non-Gaussian, converging to the Tracy-Widom distributions for large β. For $\alpha > 1/4$ the disorder is weak. For all $\alpha > 0$ we are able to identify the joint limiting distribution of the free energy and the polymer end point with a new, universal limiting density $C_\beta \exp\{A_\beta(x) - x^2/2\}$. There is an explicit asymptotic formula for the distribution of the point-to-point free energy at $\alpha = 1/4$, as well as for the rescaled polymer end point. We believe these constructions will provide a valuable tool for studying anomalous fluctuations in $(1 + 1)$ dimensions.

The research of all authors is supported by the Natural Sciences and Engineering Research Council of Canada.

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