

Relaxation of a High-Energy Quasiparticle in a One-Dimensional Bose Gas

Shina Tan,¹ Michael Pustilnik,² and Leonid I. Glazman¹

¹*Department of Physics, Yale University, New Haven, Connecticut 06520, USA*

²*School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA*

(Received 2 February 2010; published 26 August 2010)

We evaluate the relaxation rate of high-energy quasiparticles in a weakly interacting one-dimensional Bose gas. Unlike in higher dimensions, the rate is a nonmonotonic function of temperature, with a maximum at the crossover to the state of suppressed density fluctuations. At the maximum, the relaxation rate may significantly exceed its zero-temperature value. We also find the dependence of the differential inelastic scattering rate on the transferred energy. This rate yields information about temperature dependence of local pair correlations.

DOI: 10.1103/PhysRevLett.105.090404

PACS numbers: 03.75.Kk, 05.30.Jp

Recent experiments with ultracold atomic gases [1,2] have renewed interest in the fundamental properties of the elementary excitations in interacting Bose systems.

A three-dimensional (3D) Bose gas undergoes the Bose-Einstein condensation (BEC) phase transition at a sufficiently low temperature [3]. The transition affects dramatically the spectrum of elementary excitations (quasiparticles) of the system. In the Bose-condensed phase, the quasiparticles obey the Bogoliubov dispersion relation $\epsilon_q = sq\sqrt{1 + (q/2ms)^2}$, which interpolates between a phononlike linear spectrum at small momenta (here s is sound velocity and m is each boson's mass) and a free-particle-like spectrum at large momenta [3].

The BEC transition affects strongly the lifetime of low-energy quasiparticles. The relaxation rate Γ_q of quasiparticles in the phonon part of the spectrum is very sensitive to both their momenta q [4] and temperature T [5–7]: $\Gamma_q \propto \max\{(\epsilon_q)^5, \epsilon_q T^4\}$. However, the relaxation rate of high-energy quasiparticles is dominated by collisions with large momentum transfer, does not depend on either q or T , and thus is not sensitive to BEC transition [4]. Some of these long-standing predictions have been recently verified experimentally; see [1,2] for a review.

Unlike its 3D counterpart, the one-dimensional (1D) interacting Bose gas turns at low temperatures to a quasicondensate in which the long-range order is destroyed by quantum fluctuations [3,6], and the BEC transition turns to a crossover. Yet, despite this difference, the spectrum of elementary excitations in 1D is still described very well by the Bogoliubov dispersion relation [8].

However, the quasiparticle lifetime in 1D is very different from that in higher dimensions and is not as well understood. The reason is that, due to the constraints imposed by the energy and momentum conservation, two-particle collisions do not lead to a relaxation in 1D. At the same time, realizations of 1D Bose systems with cold atoms confined in tight atomic waveguides [1] are described rather well [9] by a model of bosons with zero-range repulsive interaction (the Lieb-Liniger model),

which is integrable [8,10]. In this model, the redistribution of the momenta between particles in a collision and, therefore, relaxation are absent [10]. Such apparent lack of relaxation was recently demonstrated experimentally [11] (see the discussion below).

The leading corrections to the Lieb-Liniger model have the form of a local three-particle interaction term [12,13], which breaks the integrability and brings about the quasiparticle relaxation. In this Letter, we study relaxation of a particle with a large momentum. This problem was considered recently in Ref. [13], where the inelastic relaxation rate due to three-particle collisions was evaluated in the approximation that neglects two-body repulsion. The results of Ref. [13] suggest that, very much like in 3D, the relaxation rate of high-energy quasiparticles is independent of momentum and temperature. However, in the present Letter, we demonstrate that, in a dramatic departure from the behavior in higher dimensions, the relaxation rate in 1D depends strongly on temperature even at large momenta. It has a pronounced peak at the crossover to the quasicondensate state.

We evaluate the differential and the total relaxation rates. Both can be inferred from observations of colliding clouds of cold atoms [11,14].

To describe the relaxation in a weakly interacting 1D Bose gas, we consider the simplest Hamiltonian

$$H = H_0 + V, \quad (1)$$

where

$$H_0 = \int dx \psi^\dagger(x) \left(-\frac{1}{2m} \frac{d^2}{dx^2} \right) \psi(x) + \frac{c}{2} \int dx : \rho^2(x) : \quad (2)$$

describes 1D bosons with a repulsive contact interaction (hereinafter we set $k_B = \hbar = 1$), and

$$V = -\frac{\alpha}{9m} \int dx : \rho^3(x) : \quad (3)$$

represents the leading integrability-breaking perturbation [12,13]. In Eqs. (2) and (3), $\rho(x) = \psi^\dagger(x)\psi(x)$ is the local

density operator and the colons denote the normal ordering. The strength of the interaction [represented by the second term in Eq. (2)] is characterized [8] by the dimensionless parameter $\gamma = mc/n$, where n is the 1D concentration. A finite three-particle scattering amplitude appears already in the first order in $\alpha \ll 1$.

In this Letter, we study relaxation of a boson with momentum q (we assume that $q > 0$) and kinetic energy $\xi_q = q^2/2m$, which is large compared to both temperature T and a typical interaction energy per particle ω_s :

$$\xi_q \gg \max\{T, \omega_s\}, \quad \omega_s = ms^2/2. \quad (4)$$

In the limit of a weak interaction $\gamma \ll 1$, which we consider from now on, the sound velocity s in Eq. (4) is given by [8] $s = (n/m)\sqrt{\gamma}$. The condition (4) ensures that the particle is added to an almost empty single-particle state: $f_q = \langle \psi_q^\dagger \psi_q \rangle \ll 1$ [15,16].

In the lowest (second) order in α , the differential rate of inelastic scattering is given by

$$\sigma_q(\omega) = \frac{\alpha^2}{2\pi m^2} \int_{-\infty}^{q/3} dp \delta(\omega - \xi_q + \xi_{q-p}) \mathcal{G}(p, \omega), \quad (5)$$

where $\mathcal{G}(p, \omega) = \int dx dt e^{i\omega t - ipx} \mathcal{G}(x, t)$ is the Fourier transform of the correlation function

$$\mathcal{G}(x, t) = \langle : \rho^2(x, t) :: \rho^2(0, 0) : \rangle, \quad (6)$$

which should be evaluated for the Lieb-Liniger model Eq. (2). In writing Eq. (5) we took into account the kinematic constraint $p < q/3$ on the momentum transfer in the course of three-particle scattering. The constraint translates into a restriction on the transferred energy: $\sigma_q(\omega)$ vanishes for $\omega > 5\xi_q/9$. In terms of $\sigma_q(\omega)$, the total relaxation rate is given by

$$\Gamma_q = \int d\omega \sigma_q(\omega). \quad (7)$$

The differential rate (5) at large energy transfer ω is determined by the behavior of $\mathcal{G}(x, t)$ at $t \rightarrow 0$:

$$\mathcal{G}(x, t) = 2n^2 g_2 \Psi^2, \quad \Psi(x, t) = \left(\frac{m}{2\pi i t}\right)^{1/2} e^{imx^2/2t}. \quad (8)$$

Here $n^2 g_2 = \langle : \rho^2(0, 0) : \rangle$ is the probability of finding two bosons at point $x = 0$ at time $t = 0$, and $\Psi(x, t)$ is the solution of the single-particle Schrödinger equation with the initial condition $\Psi(x, 0) = \delta(x)$. Interactions do not affect the time evolution in Eq. (8) as long as $|t| \ll \min\{1/\omega_s, 1/T\}$. Instead, the dependence on temperature and on the interaction strength enters Eq. (8) via the normalized local pair correlation g_2 . For the Lieb-Liniger model this quantity can be evaluated exactly [17]. For a weak interaction g_2 increases monotonically with T from $g_2 = 1$ at $T \ll T_s$ to $g_2 = 2$ at $T \gg T_s$ [17], where we introduced a characteristic temperature scale

$$T_s = \sqrt{\omega_s T_0} = ns; \quad (9)$$

here $T_0 = 2n^2/m$ is the quantum degeneracy temperature. (Note that $\omega_s \ll T_s \ll T_0$ for a weak interaction.)

Substitution of Eq. (8) into Eq. (5) yields the differential rate at large positive energy transfer [18]:

$$\sigma_q(\omega) = \frac{\alpha^2 T_0 g_2}{2\pi \sqrt{\xi_q \omega}} \left[\frac{1 + \sqrt{1 - \omega/\xi_q}}{(1 - \omega/\xi_q)(1 + 3\sqrt{1 - \omega/\xi_q})} \right]^{1/2}. \quad (10)$$

Equation (10) is applicable at $\max\{\omega_s, T\} \ll \omega < 5\xi_q/9$. Away from the upper end of this interval, at $\omega \ll \xi_q$, Eq. (10) reduces to

$$\sigma_q(\omega) = \frac{\alpha^2 T_0 g_2}{2\pi \sqrt{2\xi_q \omega}}. \quad (11)$$

To further analyze $\sigma_q(\omega)$ at $|\omega| \ll \xi_q$, we note that in this range of ω the momenta p contributing to the integral in Eq. (5) are small ($|p| \ll q$), and it simplifies to

$$\sigma_q(\omega) = \frac{\alpha^2}{2\pi m q} \mathcal{G}(0, \omega). \quad (12)$$

It follows from the properties of $\mathcal{G}(p, \omega)$ that the differential rate (12) satisfies the detailed balance condition

$$\sigma_q(-\omega) = e^{-\omega/T} \sigma_q(\omega). \quad (13)$$

Equation (13) implies that, while $\sigma_q(\omega) \approx \sigma_q(-\omega)$ at small energy transfers $|\omega| \ll T$, the differential rate is exponentially small at large negative ω .

To gain further understanding of the differential rate at small momentum transfer, we consider first the regime of relatively high temperatures $T \gg T_s$, when the interaction in Eq. (2) can be neglected (except for very tiny energy transfers; see below). The correlation function in Eq. (12) is then easily evaluated, resulting in

$$\begin{aligned} \sigma_q(\omega) &= \frac{\alpha^2}{2\pi^3 m q} \int \prod_{i=1}^4 dk_i f_{k_1} f_{k_2} (f_{k_3} + 1)(f_{k_4} + 1) \\ &\times \delta(k_1 + k_2 - k_3 - k_4) \\ &\times \delta(\xi_{k_1} + \xi_{k_2} - \xi_{k_3} - \xi_{k_4} + \omega), \end{aligned} \quad (14)$$

where f_k is the Bose distribution. [Equation (14) can also be derived by using Fermi's golden rule.]

At $T_s \ll T \ll T_0$ the chemical potential is given by

$$\mu = -\mu_0, \quad \mu_0 = T^2/T_0 \ll T. \quad (15)$$

At $|\omega| \ll T$ the differential rate is dominated by processes in which both the initial and the final states of the two low-energy particles involved in a collision belong to the part of the spectrum with high occupation numbers: $f_{k_i} \approx f_{k_i} + 1 \approx T/(\xi_{k_i} + \mu_0) \gg 1$. Evaluation of Eq. (14) with this approximation results in

$$\sigma_q(\omega) = \alpha^2 (T_0/\xi_q)^{1/2} (T_0/T)^3 F(|\omega|/\mu_0), \quad |\omega| \ll T. \quad (16)$$

The analytical expression for the function $F(z)$ is somewhat cumbersome. It is a monotonic function normalized as $\int_0^\infty F(z)dz = 1/8$, with a power-law behavior at $z \gg 1$, $F(z) = (2\sqrt{2}/\pi)z^{-5/2}$, and a logarithmic asymptote $F(z) = (5/16\pi^2)\ln(8e^{-7/5}/z)$ at $z \rightarrow 0$.

The logarithmic divergence at $\omega \rightarrow 0$ in Eq. (16) comes from $k_1 \approx k_2 \approx k_3 \approx k_4$ in the integral over momenta in Eq. (14) and is an artifact of the free-boson approximation. The probability of scattering two bosons with close momenta $k_1 \approx k_2$ in the initial state is suppressed at $|k_1 - k_2| \ll mc$. (There is a similar suppression for the final states $k_{3,4}$.) The logarithmic divergence in $\sigma_q(\omega)$ is thus regularized at $|\omega| \lesssim \omega_s^2/T_0$ for $T \gg T_s$.

At $T_s \ll T \ll T_0$ and $\omega \gg \mu_0$, the main contribution to the integral in Eq. (14) comes from $|k_{1,2}| \lesssim \sqrt{m\mu_0}$ and $|k_{3,4}| \sim \sqrt{m\omega} \gg |k_{1,2}|$. Neglecting $k_{1,2}$ and $\xi_{k_{1,2}}$ in the arguments of the delta functions in Eq. (14), we find

$$\sigma_q(\omega) = \frac{\alpha^2 T_0}{2\pi\sqrt{2\xi_q}|\omega|} \frac{g_2}{(1 - e^{-\omega/2T})^2}, \quad |\omega| \gg \mu_0, \quad (17)$$

where $g_2 = 2$ as appropriate for $T \gg T_s$. Equation (17) extrapolates between Eqs. (11) and (16).

With lowering the temperature, Eqs. (14)–(16) become inadequate when $\mu_0(T)$ is of the order of the interaction energy per particle ω_s , i.e., at $T \sim T_s$. At $T \ll T_s$, however, the Bogoliubov approximation for the local density operator becomes applicable [19]. In this approximation, excitations of a 1D Bose liquid are essentially free phonons (Bogoliubov quasiparticles), described by the Hamiltonian $H_B = \sum_k \varepsilon_k b_k^\dagger b_k$, with $\varepsilon_k = \sqrt{\xi_k(\xi_k + 4\omega_s)}$. In terms of phonons, the density operator has the form $\rho(x) = n + \sum_{k \neq 0} (n\xi_k/L\varepsilon_k)^{1/2} (b_k + b_{-k}^\dagger) e^{ikx}$, where L is the size of the system. By using this representation, evaluation of Eq. (12) is straightforward and yields

$$\sigma_q(\omega) = \frac{\alpha^2 T_0}{64\pi\sqrt{2\omega_s\xi_q}} \left(\frac{\omega/\omega_s}{1 - e^{-\omega/2T}} \right)^2 [1 + (\omega/4\omega_s)^2]^{-1/2} \times \{1 + [1 + (\omega/4\omega_s)^2]^{1/2}\}^{-3/2}. \quad (18)$$

At $|\omega| \gg \omega_s$, Eq. (18) reduces to Eq. (17) with $g_2 = 1$ appropriate for $T \ll T_s$. In fact, Eq. (17) is valid at any $T \ll T_0$, provided that the energy transfer falls within the range $\max\{\mu_0, \omega_s\} \ll |\omega| \ll \xi_q$. For positive ω in this range, the validity of Eq. (17) is due to the fact that the interaction has negligible effect [16] on the final states of the colliding particles ($\xi_{k_3} \approx \xi_{k_4} \approx \omega/2 \gg \omega_s$). The applicability of Eq. (17) for negative ω in the above range then follows from Eq. (13).

We show the typical plots of the differential relaxation rate $\sigma_q(\omega)$ at $T \gg T_s$ and $T \ll T_s$ in Fig. 1.

We turn now to the evaluation of the total relaxation rate, Eq. (7). There are two contributions to the integral over ω in (7):

$$\Gamma_q = \Gamma_\infty + \tilde{\Gamma}_q. \quad (19)$$

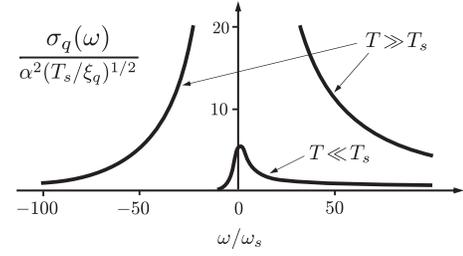


FIG. 1. The differential inelastic scattering rate $\sigma_q(\omega)$ at different temperatures (only the $|\omega| \ll \xi_q$ domain is shown). The two plots correspond to Eqs. (14) and (18), evaluated at $T = 4T_s$ and $T = T_s/4$, respectively, with $T_s = T_0/16$.

The first contribution, Γ_∞ , comes from the high-energy “tail” of $\sigma_q(\omega)$; see Eqs. (10) and (11). This contribution is independent of q and is given by [18]

$$\Gamma_\infty = \frac{\alpha^2 T_0 g_2}{3\sqrt{3}}. \quad (20)$$

Note that, unlike in higher dimensions, Γ_∞ depends on temperature via $g_2(T)$; see the discussion above.

The second contribution in Eq. (19), $\tilde{\Gamma}_q \propto \xi_q^{-1/2}$, comes from the processes with a small energy transfer $|\omega| \lesssim \max\{T, \omega_s\}$. Using Eq. (16), we find

$$\tilde{\Gamma}_q = \frac{\alpha^2 T_0}{4} \left(\frac{T_s}{\xi_q} \right)^{1/2} \left(\frac{T_0}{T_s} \right)^{3/2} \frac{T_s}{T}, \quad T_s \ll T \ll T_0. \quad (21)$$

At lower temperatures we obtain, with the help of Eq. (18),

$$\tilde{\Gamma}_q = \frac{\alpha^2 T_0}{16} \left(\frac{T_s}{\xi_q} \right)^{1/2} \left(\frac{T_0}{T_s} \right)^{3/2} \left(\frac{T}{T_s} \right)^2, \quad \frac{T_s^2}{T_0} \ll T \ll T_s. \quad (22)$$

Comparison with Eq. (20) shows that for not too large energies [$\xi_q \ll T_s(T_0/T_s)^3$] the small momentum transfer contribution $\tilde{\Gamma}_q$ dominates the relaxation rate (19) in a broad temperature interval:

$$T_s \left(\frac{\xi_q}{T_s} \right)^{1/4} \left(\frac{T_s}{T_0} \right)^{3/4} \ll T \ll T_s \left(\frac{T_s}{\xi_q} \right)^{1/2} \left(\frac{T_0}{T_s} \right)^{3/2}, \quad (23)$$

which includes $T = T_s$. At some temperature $T_{\max} \sim T_s$ within this interval, the relaxation rate reaches its peak value $\Gamma_{\max} = \Gamma_q(T_{\max})$; see Fig. 2. By extrapolating the asymptotes (21) and (22) to the region $T \sim T_s$ and finding their intersection, we estimate $T_{\max} \approx 1.6T_s$, and

$$\Gamma_{\max} \approx 0.16\alpha^2 T_0 (T_s/\xi_q)^{1/2} (T_0/T_s)^{3/2}. \quad (24)$$

The actual values of T_{\max} and Γ_{\max} may differ from the above estimates only by numerical factors; finding these values would require a systematic description of the cross-over regime $T \sim T_s$.

We now discuss briefly the feasibility of observing relaxation by inelastic collisions in a system of cold atoms confined in a cylindrical trap. In this case the effective Hamiltonian (1)–(3) can be derived explicitly, by projection onto the lowest subband of transverse quantization.

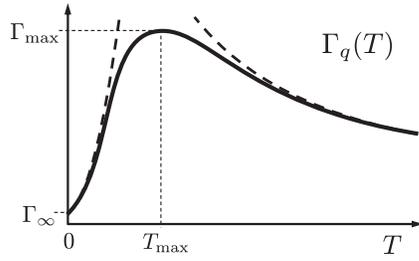


FIG. 2. Sketch of the temperature dependence of the total inelastic relaxation rate. The dependence is nonmonotonic, with a maximum at $T_{\max} \sim T_s$. The dashed lines indicate the high- and the low-temperature asymptotes [Eqs. (19)–(22)].

For a model in which the interaction in 3D is described by a pseudopotential $V_{3D}(\mathbf{r}) = 4\pi(a/m)\delta(\mathbf{r})$, where a is the s -wave scattering length [3], and with the amplitude of radial zero-point motion $a_r = (m\omega_r)^{-1/2} \gg a$ (here ω_r is the trap frequency), one finds [9,13,20]

$$\gamma = 2a/na_r^2, \quad \alpha = 18 \ln(4/3)(a/a_r)^2. \quad (25)$$

The main limitation arises due to 3-body recombination processes [21], absent in our model. The corresponding rate is $\Gamma_R = \beta n^2 g_3/a_r^4$ [21], where $g_3 = \langle \rho^3 \rangle/n^3$. Using Eqs. (20) and (25), we find [18]

$$\Gamma_\infty/\Gamma_R = \eta g_2/g_3, \quad \eta = 10.3a^4/(m\beta). \quad (26)$$

For ^{87}Rb ($a = 5.3$ nm, $\beta = 3 \times 10^{-31}$ cm⁶/s [22]), we have $\eta \approx 20$. For a weak to moderately strong interaction ($\gamma \lesssim 1$), the ratio g_2/g_3 in Eq. (26) is of the order of 1 at all T , and $\Gamma_\infty/\Gamma_R \sim 10$.

In experiments with periodically colliding clouds of cold gases [11], the 3-body recombination occurs all the time, while the scattering between the clouds takes place only during the collision itself (about one-tenth of a period in Ref. [11]). Therefore, the probability that during a period a particle participates in an inelastic collision event with a large energy transfer and the probability that it participates in a 3-body recombination process are of the same order. Accordingly, inelastic scattering with a large energy transfer is difficult to detect unambiguously.

Relaxation by the inelastic scattering with a small energy transfer is effective when the interaction is weak: $\gamma \ll 1$; indeed, the interval $T_s^2/T_0 \ll T \ll T_0$ [see Eqs. (21) and (22)] disappears for large γ . For the peak value of Γ_q [see Eq. (24)], we find

$$\Gamma_{\max}/\Gamma_R \sim 2.3\eta(T_s/\xi_q)^{1/2}\gamma^{-3/4}, \quad (27)$$

which for a fixed ratio ξ_q/T_s diverges in the limit $\gamma \rightarrow 0$.

The maximum of Γ_q is reached at $T \sim T_s$. The condition of the observability of the inelastic relaxation ($\Gamma_{\max} \gg \Gamma_R$) and the condition for the high-energy quasiparticle to be outside the quasicondensate yet well within the lowest subband of transverse quantization ($T_s \ll \xi_q \ll \omega_r$) can be satisfied simultaneously. For example, for ^{87}Rb the trap frequency $\omega_r/2\pi = 15$ kHz and concentration

$n = 7 \mu\text{m}^{-1}$ correspond to $\gamma = 0.2$ and $T_s = 120$ nK. For $\xi_q/T_s = \omega_r/\xi_q = 2.4$, Eq. (27) then yields $\Gamma_{\max}/\Gamma_R \sim 100$. The parameters above are realistic with today's experimental technology [11,23].

In conclusion, we evaluated the quasiparticle relaxation rate in a weakly interacting 1D Bose liquid. Unlike in 3D, the rate is strongly momentum- and temperature-dependent, with a maximum at $T \sim ns$, where s is the sound velocity at $T = 0$. Our predictions can be verified in experiments with colliding clouds of cold atoms.

We acknowledge discussions with V. Cheianov, D. Gangardt, R. Hulet, A. Kamenev, A. Kuzmich, G. Shlyapnikov, and D. Weiss, hospitality of Kavli Institute for Theoretical Physics at UCSB and Aspen Center for Physics, and support by the DOE (Grant No. DE-FG02-06ER46311) and by the NSF (Grant No. DMR-0906498).

- [1] I. Bloch, J. Dalibard, and W. Zwerger, *Rev. Mod. Phys.* **80**, 885 (2008).
- [2] R. Ozeri, N. Katz, J. Steinhauer, and N. Davidson, *Rev. Mod. Phys.* **77**, 187 (2005).
- [3] L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon, Oxford, 2003).
- [4] S. T. Beliaev, *Sov. Phys. JETP* **7**, 299 (1958).
- [5] P. C. Hohenberg and P. C. Martin, *Ann. Phys. (N.Y.)* **34**, 291 (1965); A. Andreev and I. M. Khalatnikov, *Sov. Phys. JETP* **17**, 299 (1963).
- [6] V. N. Popov, *Theor. Math. Phys.* **11**, 565 (1972).
- [7] L. P. Pitaevskii and S. Stringari, *Phys. Lett. A* **235**, 398 (1997); S. Giorgini, *Phys. Rev. A* **57**, 2949 (1998).
- [8] E. H. Lieb and W. Liniger, *Phys. Rev.* **130**, 1605 (1963); E. H. Lieb, *Phys. Rev.* **130**, 1616 (1963).
- [9] M. Olshanii, *Phys. Rev. Lett.* **81**, 938 (1998).
- [10] J. B. McGuire, *J. Math. Phys. (N.Y.)* **5**, 622 (1964); H. B. Thacker, *Phys. Rev. D* **11**, 838 (1975).
- [11] T. Kinoshita, T. Wenger, and D. S. Weiss, *Nature (London)* **440**, 900 (2006).
- [12] A. Muryshev *et al.*, *Phys. Rev. Lett.* **89**, 110401 (2002).
- [13] I. E. Mazets, T. Schumm, and J. Schmiedmayer, *Phys. Rev. Lett.* **100**, 210403 (2008).
- [14] E. E. Rowen *et al.*, *Phys. Rev. A* **77**, 033602 (2008).
- [15] M. Olshanii and V. Dunjko, *Phys. Rev. Lett.* **91**, 090401 (2003).
- [16] For a weak interaction, the momentum distribution at $\xi_k \gg \max\{\omega_s, T\}$ can be obtained perturbatively: $f_k = dE_2/d\xi_k \sim (nc/\xi_k)^2 \sim (\omega_s/\xi_k)^2$, where E_2 is the second-order correction to the ground state energy.
- [17] K. V. Kheruntsyan *et al.*, *Phys. Rev. Lett.* **91**, 040403 (2003).
- [18] Equations (10), (20), and (26) are valid for all γ , not just $\gamma \ll 1$.
- [19] C. Mora and Y. Castin, *Phys. Rev. A* **67**, 053615 (2003).
- [20] Our result for α is 4 times smaller than that of Ref. [13].
- [21] D. M. Gangardt and G. V. Shlyapnikov, *Phys. Rev. Lett.* **90**, 010401 (2003).
- [22] B. Laburthe Tolra *et al.*, *Phys. Rev. Lett.* **92**, 190401 (2004).
- [23] S. Hofferberth *et al.*, *Nature (London)* **449**, 324 (2007); *Nature Phys.* **4**, 489 (2008).