Casimir Repulsion between Metallic Objects in Vacuum

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(Received 19 March 2010; published 26 August 2010)

We give an example of a geometry in which two metallic objects in vacuum experience a repulsive Casimir force. The geometry consists of an elongated metal particle centered above a metal plate with a hole. We prove that this geometry has a repulsive regime using a symmetry argument and confirm it with numerical calculations for both perfect and realistic metals. The system does not support stable levitation, as the particle is unstable to displacements away from the symmetry axis.

DOI: 10.1103/PhysRevLett.105.090403 PACS numbers: 03.70.+k, 03.65.−w, 12.20.−m

Introduction.—The Casimir force between two parallel metal plates in vacuum is always attractive. A longstanding question is whether this is generally true for metallic or dielectric objects in vacuum, or whether the sign of the force can be changed by geometry alone. More precisely, can the force between noninterleaved metallic or dielectric bodies in vacuum—that is, bodies that lie on opposite sides of an imaginary separating plane—ever be repulsive? In this Letter, we answer this question in the affirmative by showing that a small elongated metal particle centered above a thin metal plate with a hole, depicted in Fig. 1(a), is repelled from the plate in vacuum when the particle is close to the plate. The particle is unstable to displacements away from the symmetry axis, so that the system does not support stable levitation, consistent with the theorem of Ref. [1]. We establish our result using a symmetry argument for an idealized case and by brute-force numerical calculations for more realistic geometries and materials. We also show that this geometry is closely related to an unusual electrostatic system in which a neutral metallic object *repels* an electric dipole (in fact, one can even obtain electrostatic repulsion for the case of a point charge [2]). Anisotropic particles are essential here; a spherical particle above a perforated plate is always attracted, although nonmonotonic effects in an isotropic case have been suggested for the null-energy condition rather than the Casimir energy [3].

Casimir repulsion is known to be impossible for $1d$/multilayer [4] or mirror-symmetric [5] metallic or dielectric geometries in vacuum. Interleaved "zipper" geometries can combine attractive interactions to yield a separating "repulsive" force [6], but the sign of the force is ambiguous in such geometries. (In contrast, in this paper the objects lie on opposite sides of a separating $z = 0^+$ plane and the interaction is unambiguously repulsive.) Repulsive forces also arise for fluid-separated geometries [7] or magnetic [8,9] or magnetoelectric materials [10,11]. A repulsive Casimir pressure was predicted within a hollow metallic sphere [12,13], but this is controversial as it does not correspond to a rigid-body motion, is intrinsically cutoff dependent [14], and the repulsion disappears if the sphere is cut in half [5]. Another proposal is to use "metamaterials" formed of metals and dielectrics arranged into complex microstructures [10,15–17]. However, no specific metamaterial geometries that exhibit Casimir repulsion have been proposed, and the theoretical result [1] indicates that repulsion in the metamaterial limit (separations $\gg$ microstructure) is impossible for parallel plate geometries.

Symmetry argument.—We begin by establishing repulsion in an idealized geometry: an infinitesimal particle centered above an infinitesimally thin perfect-metal plate with a hole. We assume the particle is electrically polarizable only in the z direction and is not magnetically polarizable at all (the limit of an infinitesimal metallic "needle") and the plate lies in the $z = 0$ plane [Fig. 1(a)]. The Casimir(-Polder) energy for such a particle at position $\mathbf{x}$ is given by [18]

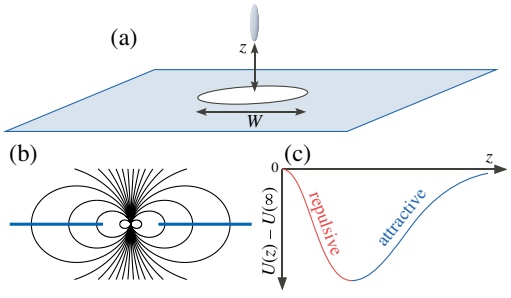

FIG. 1 (color online). (a) Schematic geometry achieving Casimir repulsion: an elongated metal particle above a thin metal plate with a hole. The idealized version is the limit of an infinitesimal particle polarizable only in the z direction. (b) At $z = 0$, vacuum-dipole field lines are perpendicular to the plate by symmetry, and so dipole fluctuations are unaffected by the plate (for *any* ω, shown here for $\omega = 0$). (c) Schematic particle-plate interaction energy $U(z) - U(\infty)$: zero at $z = 0$ and at $z \to \infty$, and attractive for $z \gg W$, so there must be Casimir repulsion (negative slope) close to the plate.

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$$U(\mathbf{x}) = -\frac{1}{2\pi}\int_0^\infty \alpha_{zz}(i\xi)\langle E_z(\mathbf{x})E_z(\mathbf{x})\rangle_{i\xi}\, d\xi. \quad (1)$$

Here α_{zz} is the electric polarizability of the particle in the z direction and $\langle E_z E_z\rangle_{i\xi}$ is the mean-square z component of the electric-field fluctuations at position $\mathbf{x}$ and imaginary frequency $\omega = i\xi$. This expectation value is evaluated in a geometry without the particle (i.e., a geometry consisting of only the perforated plate in vacuum). Conventionally, it is renormalized by subtracting the (formally infinite) mean-square fluctuations in vacuum. One way to compute the expectation value is to note that it is related to a classical electromagnetic Green's function via the fluctuation-dissipation theorem. More specifically, $\langle E_z(\mathbf{x})E_z(\mathbf{x}')\rangle_\omega$ is proportional to the electric field $E_z(\mathbf{x}')e^{-i\omega t}$ produced by an oscillating z-directed electric dipole $\mathbf{p} = p_z\hat{\mathbf{z}}e^{-i\omega t}$ at position $\mathbf{x}$.

The key idea for establishing repulsion is simple: we find a point $\mathbf{x}$ such that the classical field of an oscillating z-directed electric dipole at $\mathbf{x}$ is unaffected by the presence of the metallic plate with a hole. It then follows that $U(\mathbf{x}) = U(\infty)$, implying that the energy U must vary non-monotonically between $\mathbf{x}$ and ∞ and hence must be repulsive at some intermediate points. While in most geometries no such $\mathbf{x}$ exists, in the perforated plate geometry this condition is achieved by symmetry at $\mathbf{x} = 0$. If a z-directed electric dipole is placed at $z = 0$ in the hole, then the electric-field lines of the dipole in vacuum are *already* perpendicular to the plate by symmetry, as illustrated in Fig. 1(b); thus, the *vacuum* dipole field solves Maxwell's equations with the correct boundary conditions in the presence of the plate, and $U(z = 0) = U(\infty)$. Note that this is true by symmetry at *every* frequency ω (real or imaginary), because the dipole moment $\mathbf{p}$ at $z = 0$ is antisymmetric with respect to the $z = 0$ mirror plane. Intuitively, the basic point is that the electric dipole fluctuations of the particle do not couple to the plate at all when $z = 0$.

For large z—that is, z much larger than the hole diameter W—the presence of the hole in the plate is negligible, and we must have the usual attractive Casimir-Polder interaction. So, as schematically depicted in Fig. 1(c), we expect the interaction energy $U(z) - U(\infty)$ to be zero at $z = 0$, decrease to negative values for small $z > 0$ (leading to a repulsive force) then increase to zero for large z (leading to an attractive force). If the hole is circular, then by symmetry the force is purely in the z direction and the point of minimum U is an equilibrium position, stable under z perturbations; however, both the theorem of Ref. [1] and explicit calculations show that this equilibrium point is unstable under lateral (xy) perturbations of the particle position. In fact, numerical calculations (not shown) indicate that the particle is unstable to lateral perturbations and tilting at all separations z.

Electrostatics.—Strictly speaking, this symmetry argument only shows that the force must be repulsive at *some* $z \neq 0$: U could conceivably have multiple oscillations. To definitively rule out this possibility, we rely on the explicit numerical calculations described below. However, on an intuitive level, the basic behavior of the force can be understood from electrostatic considerations.

To see this, let us focus on the $\omega = i\xi = 0$ contribution to the Casimir energy (1); we expect the contribution from nonzero imaginary frequencies to be qualitatively similar (though this expectation can sometimes be violated, as in the inset of Fig. 4). The $\omega = 0$ contribution is proportional to the electrostatic energy of a z-directed electric dipole in the presence of a neutral metal plate. By the same arguments as above, such an electrostatic dipole must be repelled from the plate for some $z > 0$. To see this explicitly, suppose there is a static dipole at some position (0,0, z), and consider the induced charges on the plate. In the limit where the plate is infinitesimally thin, we can combine the charges on the two sides of the plate into a single surface charge density σ. On a qualitative level, we expect this total charge density to be of the form shown in Fig. 2(a), with σ positive for small r and negative for large r. In particular, σ can be constructed out of a superposition of dipoles in the $z = 0$ plane, oriented radially inward about the z axis. A simple calculation shows that vertical force on a dipole at (0,0, z) from a horizontal dipole at distance r from the z axis is repulsive if $r > 2z$ and attractive if $r < 2z$. Thus, if the hole is circular with diameter W and $z < W/4$, all the dipoles will exert a repulsive force and the total force is necessarily repulsive. On the other hand, when $z \gg W$, most of the dipoles will exert an attractive force, so the total force is attractive. In contrast, a dipole oriented *parallel* to the plate is always attracted, as one can see from the induced charge density schematically shown in Fig. 2(b). This explains why an elongated shape is necessary for the repulsive effect (see Fig. 4): dipole fluctuations parallel to the plate give rise to an attractive Casimir force.

We can confirm this picture by solving this electrostatics problem exactly in the two-dimensional ($2d$) case, where the metal plate with a hole is replaced by a metal line with a

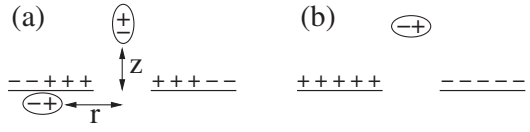

FIG. 2. (a) Schematic electrostatic interaction of a dipole with a neutral perforated plate (side view), depicting the charge density σ induced on the plate. Since σ is positive for small r and negative for large r, σ can be constructed out of a superposition of dipoles in the $z = 0$ plane, oriented radially inward about the z axis. A simple calculation then shows that the interaction is repulsive for small z. (b) In contrast, a dipole oriented parallel to the plate is always attracted, as one can see from the induced charge density shown above.

gap of width W. Assuming a $2d$ Coulomb force $F(r_{12}) = q_1 q_2 / r_{12}$, and a z-directed dipole moment p_z, we find

$$U_{\text{electrostatic}}(z) = -p_z^2 \frac{2z^2}{(W^2 + 4z^2)^2}. \quad (2)$$

The force is indeed repulsive for small z, with a sign change occurring at $z = W/2$. A similar calculation for a y-directed dipole yields a uniformly attractive force.

The repulsion in the z-directed case is quite unusual, even in electrostatics: in almost all cases, the electrostatic interaction between an electric dipole and a neutral metal object is attractive, not repulsive. Indeed, on an intuitive level, it seems almost inevitable that a dipole will induce a dipole moment in the metal object oriented so that the force is attractive. More rigorously, one can prove that this interaction is attractive in several different limits. For example, if a dipole is very far away ($z \to \infty$) or very close to the surface of a metal object, the interaction is always attractive. One can also prove that the force is attractive if the metal object is replaced by a dielectric material with a permittivity $\epsilon/\epsilon_0 = 1 + \delta$ where $0 < \delta \ll 1$, using a perturbative expansion in δ. Clearly, a special geometry is necessary to obtain a repulsive force in electrostatics, and arguably in Casimir interactions as well by extension to $\omega \neq 0$ along the imaginary frequency axis (see concluding remarks below).

Numerical demonstration.—Moving beyond the idealized geometry, we expect the repulsion to be robust under small perturbations, such as finite particle size, plate thickness, and permittivity. This expectation is validated by the explicit calculations described below. We utilize two recent numerical methods, evaluating the Casimir force at zero temperature. First, we use a finite-difference time-domain (FDTD) technique that computes the Casimir stress tensor via the Green's function [19,20], with a free-software implementation [21]. Second, we use a boundary-element method (BEM) that can solve either for the stress tensor or directly for the Casimir energy or force via a path-integral expression [22].

Figure 3 shows the Casimir force for a finite-size cylindrical metal particle (20×320 nm) above a finite-thickness ($t = 20$ nm) plate with a circular hole of diameter 1 μm, considering both perfect metals and finite-permittivity gold, along with the force on a perfect-metal sphere (diameter 60 nm) for comparison. The perfect-metal results were computed with BEM; the others were computed by our FDTD technique in cylindrical coordinates, with the gold permittivity described by $\epsilon(\omega = i\xi) = 1 + \omega_p^2/\xi^2$, where $\omega_p = 1.37 \times 10^{16}$ rad/sec (the omission of the loss term, which is convenient for FDTD [19], does not significantly affect our results). The force on the sphere is always attractive, while the force on the cylindrical particle is repulsive for a center-center separation $d \lesssim 300$ nm. Because of the finite sizes, when $d < 170$ nm the tip of the particle intrudes into the hole. However, there is a range of about 130 nm for $d > 170$ nm where the force is unambiguously repulsive: the two objects lie on opposite sides of an imaginary separating plane. Similar behavior is seen for an infinitesimally thick ($t = 0$) plate (Fig. 3). Somewhat surprisingly, the finite-permittivity gold exhibits a *stronger* repulsive force than the perfect-metal case of the same geometry; this is explained below as a consequence of the finite thickness of the plate.

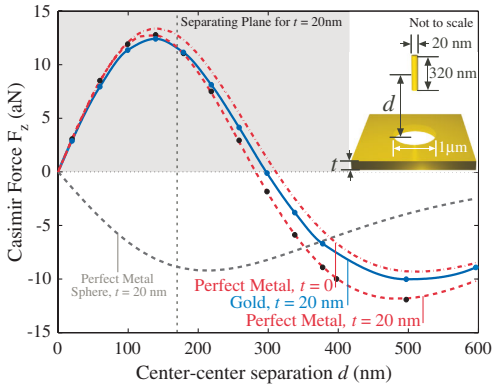

FIG. 3 (color online). Exact Casimir force for cylinder-plate geometry (inset) for perfect metals (computed with BEM) and gold (computed with FDTD); positive (shaded) force is repulsive. For $d \lesssim 300$ nm and $d > 170$ nm (vertical dashed line), the force is unambiguously repulsive as the cylinder is entirely above the plate. In contrast, a perfect-metal sphere (diameter 60 nm) is always attracted to the plate.

In order to better understand the dependence on geometry, we use BEM to explore the parameter space of a simplified $2d$ (yz) version of the problem: a metal elliptical particle above a metal line with a gap of width W. We compute the Casimir force in this setup for perfect metals and $2d$ electromagnetism, with the standard convention that the electric field is in the plane, as in a "TE" mode. (This system is equivalent to a scalar field with Neumann boundary conditions on the two objects). In Fig. 4, we explore how both the ellipticity $\gamma = a_z/a_y$ of the particle and the line thickness t affect the force, for a fixed width W and particle length $a_z = 0.002W$. As $\gamma \to 1$, the elliptical particle becomes increasing circular, and repulsion diminishes due to the attractive force associated with dipole fluctuations in the y direction. We find that the repulsive force disappears for $\gamma \lesssim 1.25$, when $t = 0$. Similarly, as t becomes larger, one can no longer make the approximation that the metal line does not affect the field of a z-directed dipole at $d = 0$, and the repulsive effect disappears by $t \approx 0.1W$ for $\gamma = 4$.

Further insight can be gained from the contribution of each imaginary frequency $\omega = i\xi$ to the force for a fixed particle-line separation $d = 0.2W$ (roughly maximum repulsion). The upper-left inset to Fig. 4 shows that as γ decreases, the attractive contributions first appear at *small* ξ, eventually making the overall force attractive. The right half of the inset shows that, in contrast, a nonzero t gives

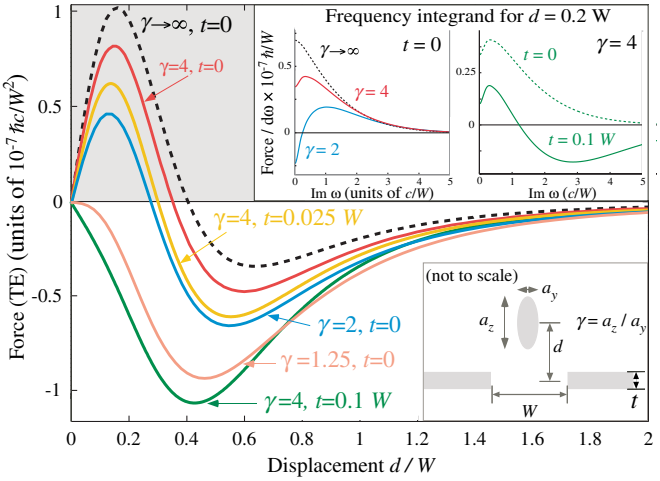

FIG. 4 (color online). Exact (BEM) $2d$ Casimir force for ellipse-line geometry (lower inset) with perfect metals and TE polarization (in-plane electric field); positive force (shaded) is repulsive. The effects of both particle width a_y and line thickness t are shown, for fixed $a_z = 0.002W$. As the ellipticity $\gamma = a_z/a_y$ decreases or t increases, the repulsive force diminishes. Upper-left inset: frequency-resolved force $F(\mathrm{Im}\omega)$ at fixed separation $d = 0.2W$ and $t = 0^+$: as γ decreases, attractive contributions arise from small $\mathrm{Im}\omega$. Upper-right inset: $F(\mathrm{Im}\omega)$ at fixed $d = 0.2W$ and $\gamma = 4$: as t increases, attractive contributions arise from large $\mathrm{Im}\omega$.

rise to attractive force contributions at *large* ξ. This may explain the larger repulsive force of real gold compared to perfect metal in Fig. 3: the finite skin depth of gold cuts off the large-ξ contributions, reducing the attractive effects of the finite plate thickness (which, in this case, dominate the attractive effects of the finite particle size and ellipticity).

Concluding remarks.—In this Letter, we have shown that the sign of the Casimir force in vacuum can be changed by geometry alone, without "cheating" by interleaving the bodies as in Ref. [6]. Consistent with Ref. [1], the geometry described here does not support stable levitation since the particle is unstable with respect to lateral (xy) translation and tilting (as shown by additional $3d$ BEM calculations). As for the question of experimental realizations, we leave this to future work, though we note that one approach would be to anchor the particle to a substrate plate made of a low permittivity material using a pillar made of the same kind of material. Assuming a periodic array of such pillars and a complementary array of holes with a unit cell of area $10\,(\mu\mathrm{m})^2$, and estimating the force per unit cell as the maximum repulsive force calculated in Fig. 3, one obtains a repulsive pressure of about 10^{-6} Pa. This is 2 or 3 orders of magnitude smaller than typical experimental sensitivities [23], but the repulsion could be increased further by shrinking or optimizing the geometry.

This geometry was motivated by the electrostatic analogue shown in Fig. 2, where a qualitatively similar effect is observed. Previously, another interesting nonmonotonic Casimir effect was also seen to have an electrostatic analogue [24]. Mathematically, the mostly nonoscillatory exponential decay of the Casimir-force contributions for imaginary frequencies $\omega = i\xi$ tends to make the total force qualitatively similar to the $\xi \to 0^+$ contribution (in fact, this similarity becomes an exact proportionality in the unretarded, van der Waals limit). This suggests that one approach for discovering "interesting" geometric Casimir effects is to first find an interesting electrostatic interaction, and then seek an analogous Casimir system.

This work was supported in part by the Harvard Society of Fellows, the U.S. DOE grant DE-FG02-97ER25308, and the DARPA contract N66001-09-1-2070-DOD.

[1] S. J. Rahi, M. Kardar, and T. Emig, Phys. Rev. Lett. **105**, 070404 (2010).
[2] M. Levin and S. G. Johnson, arXiv:1007.2175.
[3] N. Graham and K. D. Olum, Phys. Rev. D **72**, 025013 (2005).
[4] A. Lambrecht, M.-T. Jaekel, and S. Reynaud, Phys. Lett. A **225**, 188 (1997).
[5] O. Kenneth and I. Klich, Phys. Rev. Lett. **97**, 160401 (2006).
[6] A. W. Rodriguez, J. D. Joannopoulos, and S. G. Johnson, Phys. Rev. A **77**, 062107 (2008).
[7] J. Munday, F. Capasso, and V. A. Parsegia, Nature (London) **457**, 170 (2009).
[8] T. H. Boyer, Phys. Rev. A **9**, 2078 (1974).
[9] O. Kenneth, I. Klich, A. Mann, and M. Revzen, Phys. Rev. Lett. **89**, 033001 (2002).
[10] R. Zhao, J. Zhou, T. Koschny, E. N. Economou, and C. M. Soukoulis, Phys. Rev. Lett. **103**, 103602 (2009).
[11] A. G. Grushin and A. Cortijo, arXiv:1002.3481.
[12] T. H. Boyer, Phys. Rev. **174**, 1764 (1968).
[13] K. A. Milton, L. L. DeRaad, Jr., and J. Schwinger, Ann. Phys. (Leipzig) **115**, 388 (1978).
[14] R. L. Jaffe, arXiv:hep-th/0307014v2.
[15] C. Henkel and K. Joulain, Europhys. Lett. **72**, 929 (2005).
[16] U. Leonhardt and T. G. Philbin, New J. Phys. **9**, 254 (2007).
[17] F. S. S. Rosa, D. A. R. Dalvit, and P. W. Milonni, Phys. Rev. Lett. **100**, 183602 (2008).
[18] A. D. McLachlan, Proc. R. Soc. A **271**, 387 (1963).
[19] A. W. Rodriguez, A. P. McCauley, J. D. Joannopoulos, and S. G. Johnson, Phys. Rev. A **80**, 012115 (2009).
[20] A. P. McCauley, A. W. Rodriguez, J. D. Joannopoulos, and S. G. Johnson, Phys. Rev. A **81**, 012119 (2010).
[21] A. F. Oskooi *et al.*, Comput. Phys. Commun. **181**, 687 (2010).
[22] M. T. Homer Reid, A. W. Rodriguez, J. White, and S. G. Johnson, Phys. Rev. Lett. **103**, 040401 (2009).
[23] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, Rev. Mod. Phys. **81**, 1827 (2009).
[24] S. J. Rahi *et al.*, Phys. Rev. A **77**, 030101(R) (2008).