# Localization, Anomalous Diffusion, and Slow Relaxations: A Random Distance Matrix Approach

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(Received 10 February 2010; published 11 August 2010)

We study the spectral properties of a class of random matrices where the matrix elements depend exponentially on the distance between uniformly and randomly distributed points. This model arises naturally in various physical contexts, such as the diffusion of particles, slow relaxations in glasses, and scalar phonon localization. Using a combination of a renormalization group procedure and a direct moment calculation, we find the eigenvalue distribution density (i.e., the spectrum), for low densities, and the localization properties of the eigenmodes, for arbitrary dimension. Finally, we discuss the physical implications of the results.

DOI: 10.1103/PhysRevLett.105.070601 PACS numbers: 05.40.−a, 02.10.Yn, 63.50.−x, 71.23.Cq

Application of the theory of random matrices whose elements are independent Gaussian variables has proven to be rich mathematically and relevant for many physical systems [1]. In this Letter we study a different class of random matrices where the $i$, $j$th element is a function of the Euclidian distance $r_{ij}$ between pairs of points whose positions are chosen randomly and uniformly in a $d$-dimensional space. It is natural that in cases where the matrix element is related to an overlap between localized quantum-mechanical wave functions, the dependence on the distance will be exponential, i.e., $A_{ij} = e^{-r_{ij}/\xi}$, with $\xi$ being the localization length [2].

The exponential matrix is an appropriate model for various physical systems, in this Letter we will concentrate on its application to glasses relaxing to equilibrium, a particle diffusing in random environment and localization of phonons. Most of the results are derived at the low-density limit, when $\epsilon = \xi/r_{nn} \ll 1$, with $r_{nn}$ being the average nearest-neighbor distance. To understand the properties of these systems one needs to find out the distribution density $P(\lambda)$ of the eigenvalues $\lambda$ as well as the structure of the eigenmodes. An intuitive picture of the problem arises in the application to phonon localization with springs constants $K_{ij}$ that depend exponentially on the Euclidean distances between the masses; we therefore use the phonon terminology eigenmode.

The low-density limit allows us to find $P(\lambda)$ analytically employing a direct calculation of its moments, see Eq. (2) and section (2) of the supplementary material [3]. We find that $P(\lambda) \sim 1/\lambda$ in all dimensions over a broad range of $\lambda$'s. While in one dimension the normalization of $P(\lambda)$ is assured by an integrable power-law divergence at eigenvalues close to zero, for higher dimensions there is a peak related to a finite cutoff cf. Fig. 1. We use a logarithmic scale to plot $P[\log(-\lambda/2)]$ in order emphasize the deviations from the $1/\lambda$ distribution.

To comprehend the structure of the eigenmodes we use a renormalization group (RG) approach for random systems that was developed in the context of spin chains [4–7]. At each RG step, we choose the stiffest spring. Since the spring is large by construction, after finding the eigenvalue associated with the stiffest spring we can glue together the two masses at its ends creating a larger mass. At the next RG step we choose again the stiffest spring among those that remain. In this way the large eigenmodes are built initially by a pair of masses, but as the RG process progresses larger clusters of masses form eigenmodes with

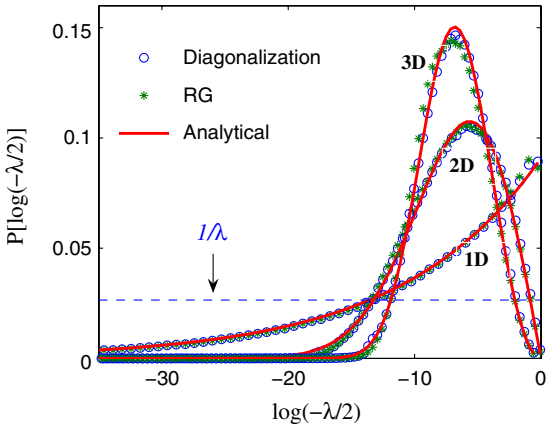


FIG. 1 (color online). Comparison between the exact numerical diagonalization (circles), and different theoretical approaches: The stars (green) show the results of a renormalization group approach, where, if one considers the problem as finding the normal modes of a spring network, the springs and masses are renormalized. The solid line (red) depicts the analytical results of Eq. (2) for 2D and 3D, and Eq. (11) of [3] for 1D. The numerical results shown are for 1, 2 and 3 dimensions, with $N = 1000$, averaged over 1000 realizations. The points were chosen in a line, square or box of side one, and $\epsilon = 0.1$, corresponding to $\xi = 10^{-4}$ in 1D, $\xi = 0.0032$ in 2D and $\xi = 0.01$ in 3D. Notice that the graph shows the distribution density of the *logarithm* of the eigenvalue, which eliminates the governing $1/\lambda$ dependence, and allows us to observe clearly the deviations from it. To demonstrate this, the horizontal dashed line (blue) shows an exact $1/\lambda$ distribution. No fitting parameters are used.

 

smaller eigenvalues. This behavior is demonstrated in Fig. 2.

The RG procedure enables us to find the average mass of a cluster at an arbitrary point along the flow, see discussion shortly before Eq. (4). Together with the exact result for the eigenvalue distribution, we can find the dependence of the cluster mass on the eigenvalue $n_c(\lambda)$ cf. Eq. (5).

We will now present in more details the properties of the exponential random matrix model, its applications, and the derivation of $P(\lambda)$ and $n_c(\lambda)$.

*Model and relevant physical problems.*—$N$ points are chosen randomly in a $d$-dimensional cube, and $r_{ij}$ is defined as the Euclidean distance between points $i$ and $j$. We define a matrix $A$, as follows: $A_{ij} = e^{-r_{ij}/\xi}$, for $i \neq j$ and $A_{ii} = -\sum_{j \neq i} A_{ij}$, the latter definition expressing a conservation law in the physical problem [8–10]. We shall be interested in determining $P(\lambda)$, the probability density of eigenvalues of the matrix $A$, for low densities (small values of $\epsilon$). Since the matrix is Hermitian, it is clear that all its eigenvalues are real, and it can also be proven that they are negative [9,11].

This model is relevant for problems from various fields of physics, but for now, we choose to focus on scalar phonon localization. It will be useful to have this problem in mind when we discuss the RG calculation.

When studying normal modes of a collection of equal masses $m$ connected by harmonic springs, one has to find the eigenmodes of a matrix $A$, where $\{A_{ij}\}$ are the spring constants, and due to momentum conservation the sum of columns vanishes. The eigenvalues are related to the frequencies by $m\omega^2 = -\lambda$, where we can choose $m$ to be unity, for convenience. The above model can be used to study phonons in a disordered lattice, where due to the randomness in the matrix elements (i.e., a distribution of spring constants), the oscillating modes may turn out to be localized in space [12]. Notice that this is a "scalar" phonon model [13], where the vectorial properties of the modes have not been taken into consideration (and thus the matrix has $N$ and not $Nd$ modes).

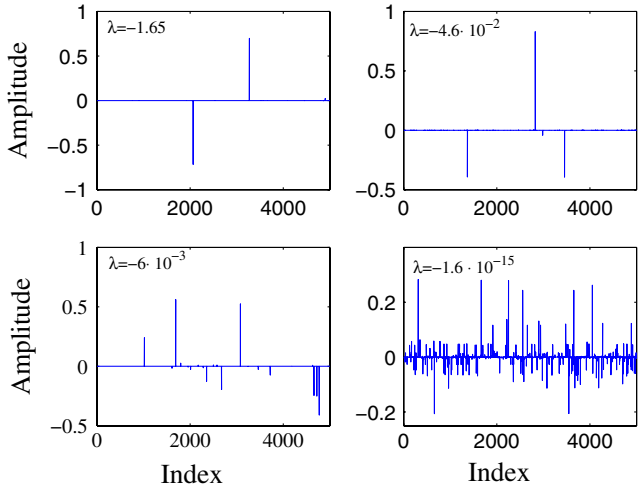


FIG. 2 (color online). Demonstration of the structure of the eigenmodes. $N = 5000$ points were chosen randomly and uniformly in a two dimensional box, with $\epsilon = \xi/r_{nn} = 0.1$. The eigenmodes become more delocalized as the eigenvalues approach zero, a condition made quantitative in Eq. (5), through the use of the RG approach. Eigenmodes are comprised of clusters of points, localized in real space.

For phonons, the density of states of the vibrations $\tilde{P}(\omega) = 2m\omega P(\lambda = -m\omega^2)$ determines the thermal properties of the system [14]. It is interesting to note that our result for $P(\lambda)$ implies $P(\omega) \sim 1/\omega$ up to corrections depending on the dimensionality.

*Derivation of the eigenvalue distribution.*—The exponential matrices model in one dimension has peculiar spectral properties [15,16]. Section (1) of the supplementary material [3] gives a simple, nonrigorous derivation for the low-density limit of the eigenvalue distribution in one dimension, which turns out to be a power law. The argument also shows that in one dimension the modes are localized, with typical size depending on $\lambda$ as a power law.

We will now develop a general formula for the eigenvalue spectrum at arbitrary dimension, in the low-density limit. We shall see that in one dimension we obtain a power law, while in higher dimension we get logarithmic corrections to a distribution $P(\lambda) \sim 1/\lambda$.

Using the following sum:

$$I_k \equiv \int P(\lambda)\lambda^k d\lambda = \frac{1}{N}\left\langle \sum_{i_1, i_2 .. i_k} A_{i_1,i_2} A_{i_2,i_3} \ldots A_{i_k,i_1} \right\rangle, \quad (1)$$

one can calculate the $k$th moment of the probability density. In [3] we find that in the low-density approximation $\epsilon \to 0$, $I_k \approx -(-2)^{k-1} d! V_d (\epsilon/k)^d$, with $V_d = \pi^{d/2}/\Gamma(d/2+1)$ the volume of a $d$-dimensional unit sphere.

Since a distribution is fully determined by its moments under certain conditions which are fulfilled in our case [17], it suffices to find a distribution which yields these moments. It can be directly checked, by performing the integrals, that the following probability density does this:

$$P(\lambda) = \frac{\epsilon^d d V_d/2[-\log(-\lambda/2)]^{d-1} e^{-V_d/2\epsilon^d(-\log(-\lambda/2))^d}}{\lambda}. \quad (2)$$

The cumulative $C(\lambda)$ of this distribution takes a particularly simple form:

$$C(\lambda) \equiv \int_\lambda^0 P(\lambda) d\lambda = e^{-(V_d/2)\epsilon^d(-\log(-\lambda/2))^d}. \quad (3)$$

In one dimension we find a power-law divergence in the distribution density, while in a dimension $d > 1$ we see from Eq. (2) that there is a finite cutoff at small eigenvalues. Figure 1 compares this formula with numerical diagonalization, for the one, two and three-dimensional cases, as well as the RG procedure. In section (2) of [3], we generalize these results to arbitrary dependence of the

matrix elements on distance, assuming sufficiently fast decay.

It is important to distinguish the above result from, for example, those of Ref. [9], which relates the eigenmodes to isolated pairs of points. There, an uncontrolled approximation was used, connecting pairs of nearest neighbors (a procedure which is not always well defined). While giving the correct qualitative dependence, it differs from Eq. (2) by a factor of 2 in the exponential, as well as in the normalization factor. As will be shown via the renormalization group approach in the next section, this difference reflects important differences in the underlying physics, and in the structure of the eigenmodes: the high frequency eigenmodes indeed consist of pairs of nearest-neighbors masses, and for them the two approaches coincide. However, the low-lying modes are comprised of an increasing number of masses, diverging as the frequency approaches zero. For any finite value of $\epsilon$, Eq. (2) might be violated for sufficiently small $\lambda$, while keeping the corrections to all moments of the distribution sufficiently small.

*Renormalization group approach.*—At *each* RG step, we choose the largest spring $K$: due to the broadness of the spring distribution, it will not be affected much by the neighboring springs, and therefore it will contribute a frequency $\omega^2 = \frac{K}{\mu} = -\lambda$, where $\mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$ is the reduced mass of the two masses it connects, $m_1$ and $m_2$. Initially, all masses are equal. Notice that mechanical intuition tells us we should choose the largest *spring* at each step, and not necessarily the largest *frequency*; i.e., the choice of the springs does not depend on the masses. Since this spring is large by construction, we can now ''glue'' the two masses together, and create a single mass $m_1 + m_2$. We also have to renormalize the springs attaching the new mass and all other masses: a reasonable way to do so is to take each of the springs between a mass $m$ and the new (joint mass) as the sum of the two springs between the mass $m$ and each of the masses $m_1$ and $m_2$. Clearly we will obtain smaller frequencies and springs and larger masses as the RG progresses.

As is shown in Fig. 1, this simple scheme captures the essential physics, with no fitting parameters. As mentioned, the reason the method works so well is the broadness of the ''spring'' distribution: for a one-dimensional case, for example, the distribution of the nearest-neighbor spring constants (which can be calculated directly from the exponential distribution of the distance intervals) follows a $P(K) \sim 1/K^{1-\epsilon}$, where $\epsilon \to 0$ in the low-density limit. Notice that for the one-dimensional case the RG procedure would choose exactly these nearest-neighbor springs, by construction. Thus, were we to neglect the mass RG we would obtain $P(\lambda) \sim 1/\lambda^{1-\epsilon}$, which recovers the low-density result mentioned earlier. In section (4) of we show how one can incorporate the mass RG to correct the one-dimensional result also for higher densities.

We will now show that using the probability density we obtained, Eq. (2), with the results of the mass RG, we can understand the localization properties of the eigenvectors. Section (3) of [3] shows how we can find the distribution of masses at a given stage of the RG process, which turns out to be approximately an exponential distribution $e^{-m/\langle m \rangle}$, with $\langle m \rangle$ changing as the RG process evolves, corresponding to the formation of larger and larger clusters. This implies that there is a typical mass $\langle m \rangle$ at each instance. Since at each step of the RG process the number of clusters decreases by one, after $k$ steps the average cluster mass is given by $\langle m \rangle = N/(N-k)$, which is also the typical size of the cluster, $n_c$. On the other hand we can find the relation between the RG step $k$ and the eigenvalue $\lambda$: In the RG process at each step the number of masses is decreased by one, and the corresponding eigenmode recorded, starting with the highest springs (and eigenvalues). At the stage of the RG flow corresponding to an eigenvalue $\lambda$, the number of masses left, $N-k$, can be calculated using Eq. (3):

$$k = N \int_{-2}^{\lambda} P(\lambda) d\lambda = N[1 - C(\lambda)]. \tag{4}$$

Combining this equation with that for $n_c = \langle m \rangle$, we find that $n_c$ depends on the eigenvalue as:

$$n_c(\lambda) \sim 1/C(\lambda) = e^{V_d/2\epsilon^d(-\log(-\lambda/2))^d}. \tag{5}$$

As we go to zero eigenvalue, the size of the eigenmodes diverges, as demonstrated in Fig. 2 on a particular example. Notice that for the case $d = 1$ we recover the power-law relation between localization length and $\lambda$ mentioned earlier, and related to Refs. [15,16].

Equation (4) also allows us to count $\mathcal{N}(n_c)$, the number of all clusters containing more than $n_c$ masses: their number is $N \int_{\lambda}^{0} P(\lambda) d\lambda = NC(\lambda)$, where we know the dependence of $\lambda$ on $n_c$ through Eq. (5). This gives $\mathcal{N}(n_c)/N = 1/n_c$.

*Physical implications.*—The mathematical model presented is relevant also for other physical problems besides phonon localization discussed above. For example, one can consider the hopping of a particle in a random environment, where $A_{ij}$ describes the transition probability of a particle from site $i$ to site $j$. If we define the probabilities of the particle to be at the different sites by a vector $\vec{p}$, then $\frac{d\vec{p}}{dt} = A\vec{p}$. In disordered systems, the transition rate often depends exponentially on the distance [18]. Another physical example of relevance, is the study of relaxations in glasses. Under certain approximations, this problem can be mapped to the study of eigenmodes of a class of random matrices related to the one described above [19]. In fact, the problem of relaxations in glasses was the original motivation for this study. In both cases of the diffusion problem and the relaxations in glasses, the Laplace transform of the distribution density plays an important role: in the hopping problem, it gives the probability to remain in

the origin [20], while for the glass relaxation it corresponds to the time dependence of the relaxation [9,21].

Upon taking the Laplace transform $\hat{P}(t)$ of the distribution density in one dimension (with argument $t$), we obtain $\hat{P}(t) \sim t^{-(\epsilon/(1+\epsilon))}$. This implies that the diffusion in this case is anomalous [18,22,23]: the probability to remain in the origin does not decrease as $1/\sqrt{t}$, as is the case for normal diffusion, but as a smaller power (the particle tends to be more localized). If we assume that after a time $t$ the particle has spread over a distance $r$, it is reasonable to assume that $\hat{p}(t)r \sim 1$, showing that $r \sim t^{\epsilon/(1+\epsilon)}$. In the case $\epsilon \to 0$, this can be approximated as $\hat{P}(t) \sim C - \log(t)$. This type of behavior has been experimentally observed in various types of glassy systems [24–26]. In higher dimensions, one obtains in the low-density limit $\hat{P}(t) = \phi[\log(t)]$, with $\phi$ a polynomial of degree $d$. This will be elaborated on in future works.

*Summary.*—We presented here a model of random matrices which captures the interesting physics of various different systems. After introducing the model, we found the eigenvalue distribution density in the low-density limit [Eq. (2)] and the localization properties of the eigenmodes [Eq. (5)] using a direct moment calculation as well as a renormalization group approach. Our results for the spectrum agree with the 1D case [15,16], and with the exact numerical diagonalization. While in one dimension it is known that for an infinite system there will be an (integrable) power-law divergence of the spectrum, in a higher dimension $d$ we found that there is a finite cutoff, where $\epsilon = \xi/r_{nn}$ is the small parameter of the theory. We used the RG approach to show that the eigenmodes are localized, and to find a relation between the spatial extent of an eigenmode and the corresponding eigenvalue, implying that this size diverges as the eigenvalue approaches zero.

We discussed the application of the model for various physical problems, such as relaxations in glasses, diffusion of particles in random media, and localization. In the future, it would be fascinating to understand also the crossover or phase transition to the high density or low disorder regime, which should present different physics.

We thank M. Aizenman, D. Cohen, M. E. Fisher, O. Hirschberg, B. Nedler, W. Schirmacher, V. Vitelli, and O. Zeitouni for important discussions. This work was supported by a BMBF DIP grant as well as by ISF and BSF grants and the Center of Excellence Program.

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