Constraints on Stable Equilibria with Fluctuation-Induced (Casimir) Forces

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We examine whether fluctuation-induced forces can lead to stable levitation. First, we analyze a collection of classical objects at finite temperature that contain fixed *and* mobile charges and show that any arrangement in space is unstable to small perturbations in position. This extends Earnshaw's theorem for electrostatics by including thermal fluctuations of internal charges. Quantum fluctuations of the electromagnetic field are responsible for Casimir or van der Waals interactions. Neglecting permeabilities, we find that any equilibrium position of items subject to such forces is also unstable if the permittivities of *all* objects are higher or lower than that of the enveloping medium, the former being the generic case for ordinary materials in vacuum.

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Earnshaw's theorem [1] states that a collection of charges cannot be held in stable equilibrium solely by electrostatic forces. The charges can attract or repel but cannot be stably levitated. While the stability of matter (due to quantum phenomena) and dramatic demonstrations of levitating frogs [2] are vivid reminders of the caveats to this theorem, it remains a powerful indicator of the constraints to stability in electrostatics. There is much current interest in forces induced by fluctuating charges (e.g., mobile ions in solution) or fluctuating electromagnetic (EM) fields (e.g., the Casimir force). The former (due to thermal fluctuations) may lead to unexpected phenomena such as attraction of like-charged macroions and is thought to be relevant to interactions of biological molecules. The latter (due mainly to quantum fluctuations) is important to the attraction (and stiction) of components of microelectromechanical devices. Here, we extend Earnshaw's theorem to some fluctuation-induced forces, thus placing strong constraints on the possibility of obtaining stable equilibria and repulsion between neutral objects.

An extension of Earnshaw's theorem [1] to polarizable objects by Braunbek [3,4] establishes that dielectric and paramagnetic ($\epsilon > 1$ and $\mu > 1$) matter cannot be stably levitated by electrostatic forces, while diamagnetic ($\mu < 1$) matter can. This is impressively demonstrated by superconductors and frogs that fly freely above magnets [2]. If the enveloping medium is not vacuum, the criteria for stability are modified by substituting the static electric permittivity ϵ_M and magnetic permeability μ_M of the medium in place of the vacuum value of 1 in the respective inequalities. In fact, if the medium itself has a dielectric constant higher than the objects ($\epsilon < \epsilon_M$), stable levitation is possible, as demonstrated for bubbles in liquids (see Ref. [5] and references therein). For dynamic fields the restrictions of electrostatics do not apply; for example, lasers can lift and hold dielectric beads [6].

We first obtain a simple extension of Earnshaw's theorem to objects containing fixed *and mobile* charges interacting via Coulomb forces. This model, depicted in Fig. 1 (left), is a classical analogue of the electrodynamic Casimir effect. The free energy is obtained, via the partition function, by integrating the positions $\{\mathbf{x}_i^J\}$ of the mobile charges $\{q_i^J\}$ over the volumes $\{\mathcal{V}_J\}$ of the corresponding objects $\{J\}$, as

$$F = -\beta^{-1} \ln \int_{\mathbf{x}_i^J \in \mathcal{V}_J} \prod_{i,J} d\mathbf{x}_i^J e^{-\beta H(\{\mathbf{x}_i^J\})}, \tag{1}$$

where $\beta = 1/(k_B T)$. Charges q_i^I and q_j^J on different objects I and J interact via the Coulomb potential $q_i^I q_j^J G_M(\mathbf{x}_i^I, \mathbf{x}_j^J)$, where $G_M(\mathbf{x}, \mathbf{x}') = (4\pi\epsilon_M|\mathbf{x} - \mathbf{x}'|)^{-1}$ is the electrostatic Green's function for a medium with per-

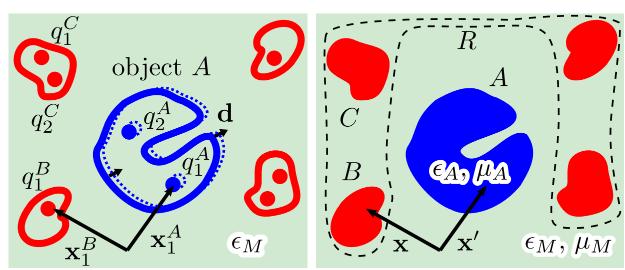

FIG. 1 (color online). Left: Each object contains mobile and stationary charges, which interact with charges in other objects according to Coulomb's law. They also interact with charges in the same object in an arbitrary manner and may be subject to an object-centered potential. The medium has static permittivity ϵ_M. The stability of the position of object A is probed by displacing it infinitesimally by vector $\mathbf{d}$. Right: Casimir energy for objects with electric permittivity $\epsilon_i(\omega, \mathbf{x})$ and magnetic permeability $\mu_i(\omega, \mathbf{x})$, embedded in a medium with uniform, isotropic, $\epsilon_M(\omega)$ and $\mu_M(\omega)$. To study the stability of object A, the rest of the objects are grouped in the combined entity R.

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mittivity ϵ_M, satisfying $-\epsilon_M \nabla^2 G_M(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$. The Hamiltonian, neglecting the kinetic energies, is thus

$$H = \sum_{I<J} \sum_{i,j} q_i^I q_j^J G_M(\mathbf{x}_i^I, \mathbf{x}_j^J) + \sum_J U_J(\{\mathbf{x}_i^J\}), \qquad (2)$$

where the summation runs over $i \in I$, $j \in J$, and $U_J(\mathbf{x}_1^J, \mathbf{x}_2^J, \ldots)$ represents the interactions among the charges and with the "container" J.

We can determine the stability of the objects' positions by examining the change in free energy under an infinitesimal shift in the position of one object, while the others are held fixed. Under a translation of object A by $\mathbf{d}$, the charges $q_1^A, \ldots, q_N^A$ in A shift to positions $\mathbf{x}_1^A + \mathbf{d}, \ldots, \mathbf{x}_N^A + \mathbf{d}$. The interaction potential $U_A(\{\mathbf{x}_i^A\})$, however, remains unchanged since the entire container A is moved. On the other hand, the Coulomb interaction between a charge q_a^A in object A and another charge q_j^J in another object J is modified to $q_a^A q_j^J G_M(\mathbf{x}_a^A + \mathbf{d}, \mathbf{x}_j^J)$. It is essential that the different objects *do not touch* to permit the infinitesimal translation of object A. The force on A is given by $-\nabla_{\mathbf{d}} F$. The position of object A is unstable if $\nabla_{\mathbf{d}}^2 F \leq 0$ and possibly stable if $\nabla_{\mathbf{d}}^2 F > 0$.

The Laplacian of the free energy is given by

$$\nabla_{\mathbf{d}}^2 F = \langle \nabla_{\mathbf{d}}^2 H \rangle - \beta[\langle (\nabla_{\mathbf{d}} H)^2 \rangle - \langle \nabla_{\mathbf{d}} H \rangle^2], \qquad (3)$$

where angular brackets denote thermal averages. The term in square brackets equals $\langle (\nabla_{\mathbf{d}} H - \langle \nabla_{\mathbf{d}} H \rangle)^2 \rangle$, which is nonnegative and makes a destabilizing contribution. The Laplacian in the first average acts only on the Green's functions in Eq. (2), which describe the interactions of the charges in object A with charges in the other objects, e.g., $\nabla_{\mathbf{d}}^2 G_M(\mathbf{x}_a^A + \mathbf{d}, \mathbf{x}_j^J)|_{\mathbf{d}=0}$; the δ functions resulting from such operations, e.g., $\delta(\mathbf{x}_a^A - \mathbf{x}_j^J)$, are always zero since the two charges lie in different volumes. Of course, the Laplacian of all other terms is zero since they do not depend on $\mathbf{d}$. Thus, the result $\nabla_{\mathbf{d}}^2 F \leq 0$ follows from the vanishing of $\nabla_{\mathbf{d}}^2 H$ for any configuration of charges within an object (as in the zero temperature Earnshaw case), and thermal fluctuations only enhance instability.

Even at zero temperature, quantum charge and current fluctuations exist, generating Casimir forces. Next, we proceed to this quantum mechanical case by considering the stability of *neutral* objects interacting via the Casimir force, which can be alternatively attributed to fluctuations of the EM field. Explicit calculations for simple geometries indicate that the direction of the force can be predicted based on the relative permittivities, and permeabilities, of the objects and the medium. By separating materials into two groups, (i) with permittivity higher than the medium or permeability lower than the medium ($\epsilon > \epsilon_M$ and $\mu \leq \mu_M$), or (ii) the other way around ($\epsilon < \epsilon_M$ and $\mu \geq \mu_M$), Casimir forces are found to be attractive between members of the same group and repulsive for different types. (While this has been shown in several examples, e.g., in Refs. [7–11], a theorem regarding the sign of the force exists only for mirror symmetric arrangements of objects [12,13].) Since ordinary materials have permittivity higher than air and permeability very close to 1, this effect causes objects to stick to one another. (The above statements will be made precise shortly.) Particularly for nanomachines, this is detrimental as the Casimir force increases rapidly with decreasing separation. This has motivated research into reversing the force; for example, a recent experiment [14] shows that, in accord with the above rules, a dielectric medium can lead to repulsion. But the sign of the force is largely a matter of perspective, since attractive forces can be easily arranged to produce repulsion along a specific direction, e.g., as in Ref. [15]. Instead, we focus on the question of stability which is more relevant to the design of microelectromechanical and levitating devices. We find that interactions between objects within the same class of material cannot produce stable configurations.

Recent theoretical advances have led to new techniques, based on scattering theory, for efficiently computing the Casimir force (see Ref. [16] for a detailed derivation and a partial review of precursors [17–19]). The exact Casimir energy of an arbitrary number of objects with linear EM response, as described in the caption of Fig. 1 (right), is given by [see Eq. (V.16) in Ref. [16]]

$$\mathcal{E} = \frac{\hbar c}{2\pi} \int_0^\infty d\kappa \, \mathrm{tr} \ln \mathbb{T}^{-1} \mathbb{T}_\infty, \qquad (4)$$

where the operator $[\mathbb{T}^{-1}(ic\kappa, \mathbf{x}, \mathbf{x}')]$ equals

$$\begin{pmatrix} [\mathbb{T}_A^{-1}(ic\kappa, \mathbf{x}_1, \mathbf{x}_1')] & [\mathbb{G}(ic\kappa, \mathbf{x}_1, \mathbf{x}_2')] & \cdots \\ [\mathbb{G}(ic\kappa, \mathbf{x}_2, \mathbf{x}_1')] & [\mathbb{T}_B^{-1}(ic\kappa, \mathbf{x}_2, \mathbf{x}_2')] & \\ \cdots & & \cdots \end{pmatrix}, \qquad (5)$$

and $\mathbb{T}_\infty$ is the inverse of $\mathbb{T}^{-1}$ with $\mathbb{G}$ set to zero. The square brackets "[ ]" denote the entire (sub)matrix with rows indicated by $\mathbf{x}$ and columns by $\mathbf{x}'$. To obtain the free energy F at finite temperature, in place of the ground state energy $\mathcal{E}$, $\int \frac{d\kappa}{2\pi}$ is replaced by the sum $\frac{kT}{\hbar c} \sum'_{\kappa_n \geq 0}$ over Matsubara "wave numbers" $\kappa_n = 2\pi n kT/\hbar c$ with the $\kappa_0 = 0$ mode weighted by 1/2. The operator $[\mathbb{T}^{-1}(ic\kappa, \mathbf{x}, \mathbf{x}')]$ has indices in position space. Each spatial index is limited to lie inside the objects $A, B, \ldots$. For both indices $\mathbf{x}$ and $\mathbf{x}'$ in the same object A, the operator is just the inverse $\mathbb{T}$ operator of that object $[\mathbb{T}_A^{-1}(ic\kappa, \mathbf{x}, \mathbf{x}')]$. For indices on different objects, $\mathbf{x}$ in A and $\mathbf{x}'$ in B, it equals the electromagnetic Green's function operator $[\mathbb{G}(ic\kappa, \mathbf{x}, \mathbf{x}')]$ for an isotropic, homogeneous medium [20]. As shown in Ref. [16], after a few manipulations, the operators $\mathbb{T}_J$ and $\mathbb{G}$ turn into the on-shell scattering amplitude matrix $\mathbb{F}_J$ of object J and the translation matrix $\mathbb{X}$, which converts wave functions between the origins of different objects. While practical computations require evaluation of the matrices in a particular wave function basis, the position space operators $\mathbb{T}_J$ and $\mathbb{G}$ are better suited to our general discussion here.

To investigate the stability of object A, we group the "rest" of the objects into a single entity R. So, $\mathbb{T}$ consists of 2×2 blocks, and the integrand in Eq. (4) reduces to $\mathrm{tr} \ln(\mathbb{I} - \mathbb{T}_A \mathbb{G} \mathbb{T}_R \mathbb{G})$. Merging the components of R poses no conceptual difficulty given that the operators are expressed in a position basis, while an actual computation of the force between A and R would remain a daunting task. If object A is moved infinitesimally by vector $\mathbf{d}$, the Laplacian of the energy is given by

$$\nabla^2_{\mathbf{d}}\mathcal{E}|_{\mathbf{d}=0} = -\frac{\hbar c}{2\pi}\int_0^\infty d\kappa \mathrm{tr}\Big[2n_M^2(ic\kappa)\kappa^2\frac{\mathbb{T}_A\mathbb{G}\mathbb{T}_R\mathbb{G}}{\mathbb{I}-\mathbb{T}_A\mathbb{G}\mathbb{T}_R\mathbb{G}} \quad (6)$$

$$+ 2\mathbb{T}_A\nabla\mathbb{G}\mathbb{T}_R(\nabla\mathbb{G})^T\frac{\mathbb{I}}{\mathbb{I}-\mathbb{T}_A\mathbb{G}\mathbb{T}_R\mathbb{G}} \quad (7)$$

$$+ 2\mathbb{T}_A\nabla\mathbb{G}\mathbb{T}_R\mathbb{G}\frac{\mathbb{I}}{\mathbb{I}-\mathbb{T}_A\mathbb{G}\mathbb{T}_R\mathbb{G}}\cdot[\mathbb{T}_A\nabla\mathbb{G}\mathbb{T}_R\mathbb{G} + \mathbb{T}_A\mathbb{G}\mathbb{T}_R(\nabla\mathbb{G})^T]\frac{\mathbb{I}}{\mathbb{I}-\mathbb{T}_A\mathbb{G}\mathbb{T}_R\mathbb{G}}\Big]. \quad (8)$$

After displacement of object A, the Green's function multiplied by $\mathbb{T}_A$ on the left and $\mathbb{T}_R$ on the right ($\mathbb{T}_A\mathbb{G}\mathbb{T}_R$) becomes $\mathbb{G}(ic\kappa, \mathbf{x}+\mathbf{d}, \mathbf{x}')$, while that multiplied by $\mathbb{T}_R$ on the left and $\mathbb{T}_A$ on the right ($\mathbb{T}_R\mathbb{G}\mathbb{T}_A$) becomes $\mathbb{G}(ic\kappa, \mathbf{x}, \mathbf{x}'+\mathbf{d})$. The two are related by transposition and indicated by $\nabla\mathbb{G}(ic\kappa, \mathbf{x}, \mathbf{x}') = \nabla_{\mathbf{d}}\mathbb{G}(ic\kappa, \mathbf{x}+\mathbf{d}, \mathbf{x}')|_{\mathbf{d}=0}$ and $[\nabla\mathbb{G}(ic\kappa, \mathbf{x}, \mathbf{x}')]^T = \nabla_{\mathbf{d}}\mathbb{G}(ic\kappa, \mathbf{x}, \mathbf{x}'+\mathbf{d})|_{\mathbf{d}=0}$ in the above equation. In the first line we have substituted $n_M^2(ic\kappa)\kappa^2\mathbb{G}$ for $\nabla^2\mathbb{G}$; the two differ only by derivatives of δ—functions which vanish since $\mathbb{G}(ic\kappa, \mathbf{x}, \mathbf{x}')$ is evaluated with $\mathbf{x}$ in one object and $\mathbf{x}'$ in another. In expressions not containing inverses of $\mathbb{T}$ operators, we can extend the domain of all operators to the entire space: $\mathbb{T}_J(ic\kappa, \mathbf{x}, \mathbf{x}') = 0$ if $\mathbf{x}$ or $\mathbf{x}'$ are not on object J and thus operator multiplication is unchanged.

To determine the signs of the various terms in $\nabla^2_{\mathbf{d}}\mathcal{E}|_{\mathbf{d}=0}$, we perform an analysis similar to Ref. [12]. However, we do not investigate convergence issues and treat the operators like matrices from the start. This means that the necessary criteria (smoothness, boundedness, compact support, etc.) are assumed to be fulfilled in realistic situations, as dealt with in Ref. [12]. The operators $\mathbb{T}_J$ and $\mathbb{G}$ are real and symmetric. An operator is positive (negative) semidefinite if all its eigenvalues are greater than or equal to zero (smaller than or equal to zero). It is easy to verify that $\mathbb{G}$ is a positive semidefinite operator, since it is diagonal in momentum space, with $\mathbb{G}(ic\kappa, \mathbf{k}) = \mu_M(ic\kappa)(\mathbb{I} + \frac{\mathbf{k}\otimes\mathbf{k}}{n_M^2(ic\kappa)\kappa^2})/[k^2 + n_M^2(ic\kappa)\kappa^2]$. If $\mathbb{M}$ is a real and symmetric matrix, it is positive semidefinite if and only if there exists a matrix $\mathbb{B}$ such that $\mathbb{M} = \mathbb{B}^T\mathbb{B}$. Let us assume that $\mathbb{T}_A$ and $\mathbb{T}_R$ are each either positive or negative semidefinite, indicated by $s^A = \pm1$ and $s^R = \pm1$, respectively. (We shall shortly show how the sign of $\mathbb{T}_J$ can be obtained from the object's permittivity and permeability.) The eigenvalues of $\mathbb{I} - \mathbb{T}_A\mathbb{G}\mathbb{T}_R\mathbb{G}$, which equal those of $\mathbb{I} - s^A\sqrt{s^A\mathbb{T}_A}\mathbb{G}\mathbb{T}_R\mathbb{G}\sqrt{s^A\mathbb{T}_A}$, are strictly positive, since the energy is real. [The above expression appears in the integrand of Eq. (4) if there are only two objects.] Under the trace we always encounter the combination $(\mathbb{I} - \mathbb{T}_A\mathbb{G}\mathbb{T}_R\mathbb{G})^{-1}\mathbb{T}_A$, which, by taking advantage of its symmetries and definite sign, can be written as $s^A\mathbb{B}^T\mathbb{B}$, where $\mathbb{B} = (\mathbb{I} - s^A\sqrt{s^A\mathbb{T}_A}\mathbb{G}\mathbb{T}_R\mathbb{G}\sqrt{s^A\mathbb{T}_A})^{-1/2}\sqrt{s^A\mathbb{T}_A}$. The first term, Eq. (6), can now be rearranged as $\mathrm{tr}s^A\mathbb{B}^T\mathbb{B}\mathbb{G}\mathbb{T}_R\mathbb{G} = s^As^R\mathrm{tr}[(\mathbb{B}\mathbb{G}\mathbb{R})(\mathbb{R}^T\mathbb{G}\mathbb{B}^T)]$ by setting $\mathbb{T}_R = s^R\mathbb{R}\mathbb{R}^T$, and its sign is s^As^R. In the same way, Eq. (7) can be recast as $s^As^R\mathrm{tr}[(\mathbb{B}\nabla\mathbb{G}\mathbb{R})\cdot(\mathbb{B}\nabla\mathbb{G}\mathbb{R})^T]$, and its sign is thus also set by s^As^R. Last, the term in Eq. (8) can be rewritten as $[\mathbb{B}\nabla\mathbb{G}\mathbb{T}_R\mathbb{G}\mathbb{B}^T + \mathbb{B}\mathbb{G}\mathbb{T}_R(\nabla\mathbb{G})^T\mathbb{B}^T]^2$. Since this is the square of a symmetric matrix, its eigenvalues are greater than or equal to zero, irrespective of the signs of $\mathbb{T}_A$ and $\mathbb{T}_R$. Overall, the Laplacian of the energy is smaller than or equal to zero as long as $s^As^R \geq 0$.

What determines the sign of $\mathbb{T}_J$? It can be related to the electrodynamic "potential" $\mathbb{V}_J$, discussed in the next paragraph, by $\mathbb{T}_J \equiv \mathbb{V}_J/(\mathbb{I} + \mathbb{G}\mathbb{V}_J)$ [16]. It is then positive or negative semidefinite depending on the sign s^J of $\mathbb{V}_J$, since $\mathbb{T}_J = s^J\sqrt{s^J\mathbb{V}_J}[\mathbb{I}/(\mathbb{I} + s^J\sqrt{s^J\mathbb{V}_J}\mathbb{G}\sqrt{s^J\mathbb{V}_J})]\sqrt{s^J\mathbb{V}_J}$. The denominator $\mathbb{I} + s^J\sqrt{s^J\mathbb{V}_J}\mathbb{G}\sqrt{s^J\mathbb{V}_J}$ is positive semidefinite, even if $s^J = -1$, as its eigenvalues are the same as $\sqrt{\mathbb{G}}(\mathbb{G}^{-1} + \mathbb{V}_J)\sqrt{\mathbb{G}}$; the term in the parentheses is just the (nonnegative) Hamiltonian of the field and the object J: $\mathbb{G}^{-1} + \mathbb{V}_J = \nabla\times\mu^{-1}(ic\kappa, \mathbf{x})\nabla\times + \mathbb{I}\kappa^2\epsilon(ic\kappa, \mathbf{x})$. Here, we have used $\mathbb{V}_J$ as given below, and $\epsilon(ic\kappa, \mathbf{x})$ and $\mu(ic\kappa, \mathbf{x})$ are the response functions defined over all space, either of object J or of the medium, depending on $\mathbf{x}$.

The analysis so far applies to each imaginary frequency $ic\kappa$. As long as the signs of $\mathbb{T}_A$ and $\mathbb{T}_R$ are the same over all frequencies, $\nabla^2_{\mathbf{d}}\mathcal{E}|_{\mathbf{d}=0}$ is proportional to $-s^As^R -$ (positive term) [21]. We are left to find the sign of the potential $\mathbb{V}_J(ic\kappa, \mathbf{x}) = \mathbb{I}\kappa^2[\epsilon_J(ic\kappa, \mathbf{x}) - \epsilon_M(ic\kappa)] + \nabla\times[\mu_J^{-1}(ic\kappa, \mathbf{x}) - \mu_M^{-1}(ic\kappa)]\nabla\times$ of the object A and the compound object R [22]. The sign is determined by the relative permittivities and permeabilities of the objects and the medium: If $\epsilon_J(ic\kappa, \mathbf{x}) > \epsilon_M(ic\kappa)$ and $\mu_J(ic\kappa, \mathbf{x}) \leq \mu_M(ic\kappa)$ hold for all $\mathbf{x}$ in object J, the potential $\mathbb{V}_J$ is positive. If the opposite inequalities are true, $\mathbb{V}_J$ is negative. The curl operators surrounding the magnetic permeability do not influence the sign, as in computing an inner product with $\mathbb{V}_J$ they act symmetrically on both sides. For vacuum $\epsilon_M = \mu_M = 1$, and material response functions $\epsilon(ic\kappa, \mathbf{x})$ and $\mu(ic\kappa, \mathbf{x})$ are analytical continuations of the permittivity and permeability, respectively, for real frequencies [23]. While $\epsilon(ic\kappa, \mathbf{x}) > 1$ for positive κ, there are no restrictions other than positivity on $\mu(ic\kappa, \mathbf{x})$. [For nonlocal and nonisotropic response, various inequalities must be generalized to the tensorial operators $\overleftrightarrow{\epsilon}(ic\kappa, \mathbf{x}, \mathbf{x}')$ and $\overleftrightarrow{\mu}(ic\kappa, \mathbf{x}, \mathbf{x}')$.]

Thus, levitation is not possible for collections of objects characterized by $\epsilon_J(ic\kappa, \mathbf{x})$ and $\mu_J(ic\kappa, \mathbf{x})$ falling into one of the two classes described earlier: (i) $\epsilon_J/\epsilon_M > 1$ and $\mu_J/\mu_M \leq 1$ (positive $\mathbb{V}_J$ and $\mathbb{T}_J$) or (ii) $\epsilon_J/\epsilon_M < 1$ and $\mu_J/\mu_M \geq 1$ (negative $\mathbb{V}_J$ and $\mathbb{T}_J$). (Under these conditions parallel slabs attract.) The frequency and space dependence of the functions has been suppressed in these inequalities. In vacuum, $\epsilon_M(ic\kappa) = \mu_M(ic\kappa) = 1$; since $\epsilon(ic\kappa, \mathbf{x}) > 1$ and the magnetic response of ordinary materials is typically negligible [23], one concludes that stable equilibria of the Casimir force do not exist. If objects A and R, however, belong to different categories—under which conditions the parallel plate force is repulsive—then the terms under the trace in Eqs. (6) and (7) are negative. The positive term in Eq. (8) is typically smaller than the first two, as it involves higher powers of $\mathbb{T}$ and $\mathbb{G}$. In this case stable equilibrium is possible, as demonstrated recently for a small inclusion within a dielectric filled cavity [24]. For the remaining two combinations of inequalities involving ϵ_J/ϵ_M and μ_J/μ_M, the sign of $\mathbb{V}_J$ cannot be determined *a priori*. But for realistic distances between objects and the corresponding frequency ranges, the magnetic susceptibility is negligible for ordinary materials, and the inequalities involving μ can be ignored.

In summary, the instability theorem applies to all cases where the coupling of the EM field to matter can be described by response functions ϵ and μ, which may vary continuously with position and frequency. Obviously, for materials which at a microscopic level cannot be described by such response functions, e.g., because of magnetoelectric coupling, our theorem is not applicable.

Even complicated arrangements of materials obeying the above conditions are subject to the instability constraint. For example, metamaterials incorporating arrays of microengineered circuitry mimic, at certain frequencies, a strong magnetic response and have been discussed as candidates for Casimir repulsion across vacuum. (References [25,26] critique repulsion from dielectric- or metallic-based metamaterials, in line with our following arguments.) In our treatment, in accord with the usual electrodynamics of macroscopic media, the materials are characterized by $\epsilon(ic\kappa, \mathbf{x})$ and $\mu(ic\kappa, \mathbf{x})$ at *mesoscopic* scales. In particular, chirality and large magnetic response in metamaterials are achieved by patterns made from ordinary metals and dielectrics with well-behaved $\epsilon(ic\kappa, \mathbf{x})$ and $\mu(ic\kappa, \mathbf{x}) \approx 1$ at *short* scales. The interesting EM responses merely appear when viewed as "effective" or "coarse-grained." Clearly, the coarse-grained response functions, which are conventionally employed to describe metamaterials, should produce, in their region of validity, the same scattering amplitudes as the detailed mesoscopic description. Consequently, as long as the metamaterial can be described by $\epsilon(ic\kappa, \mathbf{x})$ and $\mu(ic\kappa, \mathbf{x}) \approx 1$, the eigenvalues of the $\mathbb{T}$ operators are constrained as described above and, hence, subject to the instability theorem. Thus, the proposed use of chiral metamaterials in Ref. [27] cannot lead to stable equilibrium since the structures are composites of metals and dielectrics. Finally, we note that instability also excludes repulsion between two objects that obey the above conditions, if one of them is an infinite flat plate with continuous translational symmetry: Repulsion would require that the energy as a function of separation from the slab should have $\partial_d^2 \mathcal{E} > 0$ at some point since the force has to vanish at infinite separation. A metamaterial does not have continuous translational symmetry at short length scales, but this symmetry is approximately valid in the limit of large separations (long wavelengths), where the material can be effectively described as a homogeneous medium. At short separations, lateral displacements might lead to repulsion that, however, must be compatible with the absence of stable equilibrium.

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[21] $\mathbb{T}_A$ and $\mathbb{T}_R$ suffice to have the same sign over frequencies which contribute most to the integral (or sum) in Eq. (4).
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