Relationship between Complete Coherence in the Space-Time and in the Space-Frequency Domains

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We present some new results relating to properties of completely coherent optical fields. Our analysis elucidates the relationship between the theories of such fields in the space-time and in the space-frequency domains. We also show that the concept of cross-spectral purity, introduced by L. Mandel many years ago, plays an important role in clarifying this relationship.

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The invention of the laser, almost exactly half a century ago, has made it possible to generate highly coherent light beams. As is well known, such beams have found numerous applications in many fields, especially, in physics, chemistry, medicine and biology. The invention also triggered many studies of coherence properties of light.

The concepts of completely coherent light and of monochromaticity have often been incorrectly regarded as synonymous. This misconception persists, even though differences between them have been noted in several publications (see, for example, [1,2]). The electromagnetic field associated with a monochromatic light oscillates in a *deterministic* way with time. However, no optical field found in nature or produced in a laboratory is deterministic; it always undergoes some random fluctuations. There are many causes for the randomness. For light generated by thermal sources, it is mainly due to the fact that the optical field has contributions from many atoms, which radiate essentially independently of each other. Even in beams generated by good-quality lasers, random fluctuations are present, because contributions from spontaneous emission cannot be eliminated. It is well known that over sufficiently long time interval, the phase of the output of even a well stabilized laser undergoes random fluctuations (see, for example, [3], Sec. VI.7).

The distinction between complete coherence and strict monochromaticity is not just of academic interest. Very recently, practical implications of this difference have began to be appreciated; that led, for example, to the theoretical solution of a classic old problem in crystallography, namely, showing how the phases of x-ray beams diffracted by crystalline media may be determined [4,5]. The appreciation of this distinction appears to have considerable potential for developing new phase measurement techniques (see, for example, Ref. [6]).

It should be clear from these remarks that the concept of complete coherence is an important one, and evidently should be investigated more carefully than has been done up to now. This Letter is a modest contribution towards understanding of the properties of spatially completely coherent light. We investigate the implications of complete spatial coherence of an optical field on the frequency components present in its spectrum. It will become clear shortly that our analysis clarifies an important aspect of the relationship between the space-time and the space-frequency formulations of coherence theory. It also reveals that cross-spectrally pure light (see [7,8]; also, [9], Sec. 4.5) has some special properties which are relevant for understanding of the relationship.

The foundations of the coherence theory of light was laid down by Zernike in a classic paper [10] published in 1938. Zernike defined the degree of coherence of light at two points in a wave field as the maximum visibility of interference fringes formed by superposing light arriving from the two points. He also showed that the maximum value of the fringe visibility is equal to the modulus of the normalized cross-correlation function of the optical fields at the two points, considered at the same instant of time. Since Zernike's formulation did not take into account a possible time difference between the interfering beams, it could not explain some rather important coherence properties of light. In 1955, Zernike's definition of coherence was generalized by including time difference between the interfering beams and it was shown that the corresponding correlation functions satisfy rigorously certain propagation laws [11]. This formulation is known as the *space-time formulation* of the theory of optical coherence. According to this formulation, the randomly fluctuating field, at a point P with position $\mathbf{r}$, at a time t, may be represented by a statistical ensemble $\{V(\mathbf{r};t)\}$ of realizations, which for simplicity we assume to be scalar (see, for example, [12], Sec. 2.1). The second-order correlation properties of such a field in the space-time domain, at a pair of points $Q_1(\mathbf{r}_1)$, $Q_2(\mathbf{r}_2)$, may be characterized by the so-called *mutual coherence function* $\Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)$ given by (see, for example, [12], Sec. 3.1)

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) \equiv \langle V^*(\mathbf{r}_1; t)V(\mathbf{r}_2; t+\tau)\rangle. \tag{1}$$

In this expression the asterisk denotes the complex conjugate and the angular brackets denote ensemble average.

We assume that the field is statistically stationary, at least in the wide sense ([9], Sec. 2.2); consequently the expression on the right-hand side of Eq. (1) is independent of t, but depends on τ. The "space-time" degree of coherence $\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)$ at a pair of points is defined by the expression ([11]; see also, [12], Sec. 3.1)

$$\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) \equiv \frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)}{\sqrt{I(\mathbf{r}_1)}\sqrt{I(\mathbf{r}_2)}}, \tag{2}$$

where $I(\mathbf{r}) \equiv \Gamma(\mathbf{r}, \mathbf{r}; 0)$ represents the (average) intensity at the point $P(\mathbf{r})$. It can be shown that $0 \leq |\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)| \leq 1$. When $|\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)| = 1$, the field is said to be *spatially completely coherent* at the pair of points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$. The other extreme case, $\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) = 0$, represents *complete spatial incoherence* of the field at the two points.

The physical significance of the degree of coherence of light at two points can be understood from the analysis of Young's two-pinhole experiment. Suppose that a light beam is incident from the half-space $z < 0$ onto an opaque screen A, placed in the plane $z = 0$, containing two pinholes at $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$ (see Fig. 1). For the sake of simplicity, we assume that the beam is incident normally on the screen A. In general, interference fringes will be formed on a screen B, placed in a plane $z = z_0 > 0$, some distance behind the screen A (see Fig. 1). We assume that the contributions to the averaged intensity at the point $P(\mathbf{r})$ on the screen B from light emerging from each of the two pinholes are equal. One can readily show that the average intensity $I(\mathbf{r})$ at the point $P(\mathbf{r})$ is then given by the expression [see, for example, [12], Sec. 3.1, Eq. (16)]

$$I(\mathbf{r}) = 2I^{(1)}(\mathbf{r})\{1 + |\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)| \cos[\alpha(\mathbf{r}_1, \mathbf{r}_2; \tau)]\}, \tag{3}$$

where $I^{(1)}(\mathbf{r})$ is the intensity contribution at $P(\mathbf{r})$ from each of the two pinholes, $\alpha(\mathbf{r}_1, \mathbf{r}_2; \tau) = \text{Phase}\{\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)\}$, and $\tau = (R_2 - R_1)/c$, c being the speed of light in free space. One can readily show from Eq. (3) that the visibility $\mathcal{V}$ of the fringes at the point $P(\mathbf{r})$ is equal to the modulus $|\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)|$ of the spatial degree of coherence of light at the pair of points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$ (see Fig. 1) with $\tau = (R_2 - R_1)/c$ [[12], Sec. 3.1, Eq. (19)], i.e., that

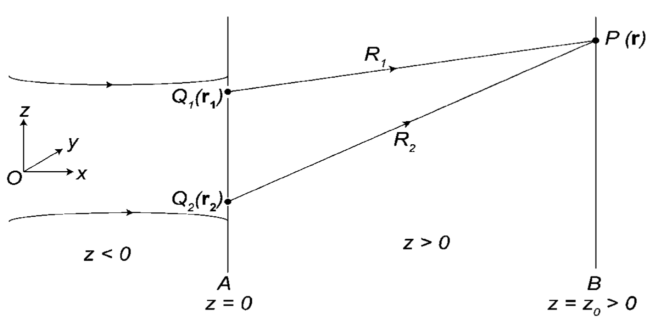

FIG. 1. Illustrating the notations.

$$\mathcal{V} \equiv \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = |\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)|, \qquad 0 \leq \mathcal{V} \leq 1. \tag{4}$$

It is evident that the more coherent the field is at the pair of points Q_1 and Q_2, the greater will be the value of the fringe visibility. The phase of the degree of coherence may also be determined from such experiment (see, for example, [9], p-167).

Let us now consider an optical field which is spatially *completely coherent* at the pair of points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$, for a particular value τ_0 of the time delay τ, i.e., that

$$|\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau_0)| = 1. \tag{5}$$

It has been shown, not long ago [13–15], that in this case the fluctuating fields at the points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$ are statistically similar, in the sense that

$$V(\mathbf{r}_2; t + \tau_0) = A_{12}V(\mathbf{r}_1; t), \tag{6}$$

where A_{12} is a time independent (generally complex) number. A_{12} is a measurable quantity, whose modulus and phase are given by the expressions [13]

$$|A_{12}| = \sqrt{\frac{I(\mathbf{r}_2)}{I(\mathbf{r}_1)}}, \tag{7a}$$

$$\text{Phase}\{A_{12}\} = \text{Phase}\{\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau_0)\}. \tag{7b}$$

The statistical similarity relation (6) can also be expressed in the form

$$V(\mathbf{r}_2; t) = A_{12}V(\mathbf{r}_1; t - \tau_0). \tag{8}$$

On using Eqs. (1), (6), and (8), the mutual coherence function of such a field may be expressed in the forms

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) = A_{12}\Gamma(\mathbf{r}_1, \mathbf{r}_1; \tau - \tau_0) = \frac{\Gamma(\mathbf{r}_2, \mathbf{r}_2; \tau - \tau_0)}{A_{12}^*}. \tag{9}$$

There is an alternative formulation of coherence theory known as the *space-frequency formulation* (see, for example, [12], Ch. 4), which turned out to be rather useful. It led to the discoveries and understanding of several physical phenomena, such as correlation-induced spectral changes [16] and changes in the polarization properties of light on propagation [17]. Recent studies have also revealed a great usefulness of this theory in connection with determining structures of objects by inverse scattering technique (see, for example, Refs. [4,5,18]). However the relationship between the space-frequency formulation and the older space-time formulation has so far not been systematically investigated [19].

In the space-frequency formulation, a statistical stationary field is represented, at each frequency ω, by the *cross-spectral density function* $W(\mathbf{r}_1, \mathbf{r}_2, \omega)$, to be abbreviated by CSDF, which is the Fourier transform of the mutual coherence function:

$$W(\mathbf{r}_1, \mathbf{r}_2, \omega) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \exp[i\omega\tau] d\tau. \quad (10)$$

$W(\mathbf{r}_1, \mathbf{r}_2, \omega)$ may be shown to be also a correlation function, i.e., that it can be represented in the form

$$W(\mathbf{r}_1, \mathbf{r}_2; \omega) = \langle U^*(\mathbf{r}_1; \omega) U(\mathbf{r}_2; \omega)\rangle_\omega, \quad (11)$$

where $U(\mathbf{r}; \omega)$ is a typical member of a suitably constructed ensemble of monochromatic realizations, all of frequency ω ([12], Sec. 4.1). The subscript ω on the angular bracket in Eq. (11) indicates that the average is taken over that ensemble. One may again consider a Young's two-pinhole experiment (Fig. 1), but restricting one's attention to only one frequency component ω of the light [20]. We assume that the contributions to the spectral density (intensity at frequency ω) at the point $P(\mathbf{r})$ from each of the pinholes are the same. The distribution of the spectral density $S(\mathbf{r}; \omega) \equiv W(\mathbf{r}, \mathbf{r}; \omega)$ on the screen B is given by the expression ([12], Sec. 4.2)

$$S(\mathbf{r}; \omega) = S^{(1)}(\mathbf{r}; \omega)\{1 + |\mu(\mathbf{r}_1, \mathbf{r}_2; \omega)| \times \cos[\beta(\mathbf{r}_1, \mathbf{r}_2; \omega) - \delta]\}, \quad (12)$$

where $S^{(1)}(\mathbf{r}; \omega)$ is the contribution from either of the two pinholes, $\delta = \omega(R_2 - R_1)/c$,

$$\mu(\mathbf{r}_1, \mathbf{r}_2; \omega) \equiv \frac{W(\mathbf{r}_1, \mathbf{r}_2; \omega)}{\sqrt{S(\mathbf{r}_1, \omega)}\sqrt{S(\mathbf{r}_2, \omega)}} \quad (13)$$

is the *spectral degree of coherence* and $\beta(\mathbf{r}_1, \mathbf{r}_2; \omega)$ is the phase of $\mu(\mathbf{r}_1, \mathbf{r}_2; \omega)$. The formula (12) is known as the *spectral intensity law*. By analogy with the space-time formulation, one can readily show that $|\mu(\mathbf{r}_1, \mathbf{r}_2; \omega)|$ is equal to the fringe visibility associated with the frequency component ω, in the experiment shown in Fig. 1. It should be noted that in this case, unlike in the case of the space-time formulation, the fringe visibility is constant over the screen B. When $|\mu(\mathbf{r}_1, \mathbf{r}_2; \omega)| = 1$, the field at the two points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$ is said to be *spectrally* completely coherent at the frequency ω. If $\mu(\mathbf{r}_1, \mathbf{r}_2; \omega) = 0$, the field is said to be *spectrally* completely incoherent at the two points, at that frequency.

We will now investigate the consequence of Eq. (9) in the space-frequency formulation. From Eqs. (9) and (10), it can readily be shown that the CSDF of such a field at the pair of points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$, at every frequency ω, is given by expressions

$$W(\mathbf{r}_1, \mathbf{r}_2; \omega) = \exp[i\omega\tau_0] A_{12} S(\mathbf{r}_1; \omega) = \exp[i\omega\tau_0]\frac{S(\mathbf{r}_2; \omega)}{A_{12}^*}. \quad (14)$$

This CSDF may be expressed in the form

$$W(\mathbf{r}_1, \mathbf{r}_2; \omega) = \exp[i\omega\tau_0]\sqrt{S(\mathbf{r}_1; \omega)}\sqrt{S(\mathbf{r}_2; \omega)}\left[\frac{A_{12}}{A_{12}^*}\right]^{1/2}. \quad (15)$$

From Eqs. (13) and (14) [or (15)] one can readily show that, under these circumstances,

$$|\mu(\mathbf{r}_1, \mathbf{r}_2; \omega)| = 1, \quad (16)$$

at *every frequency* ω present in the spectrum of the light, i.e., the field is *spectrally* completely coherent at each frequency, at the pair of points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$. We have thus established the following theorem:

Theorem.—If a statistically stationary optical field is spatially completely coherent at a pair of points, for a particular value τ_0 of the parameter τ, in the space-time formulation, then at that pair of points it is also spectrally completely coherent at every frequency ω present in the spectrum of the field.

The converse of this theorem is, however, *not true* in general; i.e., even if every frequency component of an optical field is spectrally completely coherent at a pair of points, the field may not be spatially coherent at that pair of points in the space-time formulation for any value of τ, as we will now show. To prove this assertion let us consider an optical field which is spectrally completely coherent ($|\mu(\mathbf{r}_1, \mathbf{r}_2; \omega)| = 1$), at a pair of points $Q_1(\mathbf{r}_1)$ and $Q_2(\mathbf{r}_2)$, at each frequency ω present in its spectrum. Using Eq. (13), one has, in this case,

$$W(\mathbf{r}_1, \mathbf{r}_2; \omega) = \sqrt{S(\mathbf{r}_1; \omega)}\sqrt{S(\mathbf{r}_2; \omega)} \exp[i\beta(\mathbf{r}_1, \mathbf{r}_2; \omega)], \quad (17)$$

where $\beta(\mathbf{r}_1, \mathbf{r}_2; \omega)$ is the argument (phase) of the unimodular spectral degree of coherence $\mu(\mathbf{r}_1, \mathbf{r}_2; \omega)$. On taking the Fourier transform of Eq. (17) and on using Eq. (2), one readily finds that

$$\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau) = \frac{\int_0^\infty [S(\mathbf{r}_1; \omega) S(\mathbf{r}_2; \omega)]^{1/2} e^{i\beta(\mathbf{r}_1, \mathbf{r}_2; \omega)} e^{-i\omega\tau} d\omega}{[\int_0^\infty S(\mathbf{r}_1; \omega) d\omega \quad \int_0^\infty S(\mathbf{r}_2; \omega) d\omega]^{1/2}}. \quad (18)$$

Using the Cauchy-Schwarz inequality, one can readily show from this formula that, in general, $|\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)| \neq 1$. This implies that, even if each frequency component of an optical field produces fringes of unit visibility in an Young's interference experiment [21], there may *not* be a value τ_0 of the parameter τ for which $|\gamma(\mathbf{r}_1, \mathbf{r}_2; \tau_0)| = 1$. However, we will now show that in the special case *when the field is cross-spectrally pure the converse of the theorem holds*.

We again consider a Young's two-pinhole experiment (Fig. 1). The light at the two pinholes is said to be cross-spectrally pure (see [7,8]; also, [9], Sec. 4.5), if (i) the spectral densities of the light at the two pinholes are proportional to each other at all frequencies [Eq. (19a) below] and if (ii) there exists a region around some point $P_0(\mathbf{r_0})$ in the plane of observation B where the spectral distribution of the light is proportional to the spectral distribution of light at the pinholes, at all frequencies [Eq. (19b) below], i.e., if the following relations hold for all frequencies :

$$S(\mathbf{r}_2;\omega) = C_{12} S(\mathbf{r}_1;\omega), \tag{19a}$$

$$S(\mathbf{r}_0;\omega) = D S(\mathbf{r}_1;\omega), \tag{19b}$$

where C_{12} and D are positive quantities that are independent of ω. In such a case, one can show [see, [9], Sec. 4.5, Eq. (4.5–7)] that the spectral degree of coherence may be expressed in the form

$$\mu(\mathbf{r}_1, \mathbf{r}_2;\omega) = F(\mathbf{r}_1, \mathbf{r}_2;\tau_0)\exp[i\omega\tau_0], \tag{20}$$

where $F(\mathbf{r}_1, \mathbf{r}_2;\tau_0)$ is independent of ω. In the present case, since $|\mu(\mathbf{r}_1, \mathbf{r}_2;\omega)| = 1$, Eq. (20) gives

$$\mu(\mathbf{r}_1, \mathbf{r}_2;\omega) = \exp[i\psi(\mathbf{r}_1, \mathbf{r}_2;\tau_0)]\exp[i\omega\tau_0], \tag{21}$$

where $\psi(\mathbf{r}_1, \mathbf{r}_2;\tau_0)$ is the argument (phase) of $F(\mathbf{r}_1, \mathbf{r}_2;\tau_0)$. Using Eqs. (13), (19a), and (21), one finds at once that

$$W(\mathbf{r}_1, \mathbf{r}_2;\omega) = \sqrt{C_{12}} S(\mathbf{r}_1;\omega)\exp[i\psi(\mathbf{r}_1, \mathbf{r}_2;\tau_0)] \times \exp[i\omega\tau_0]. \tag{22}$$

It is interesting to note that this form of the CSDF is equivalent to that given by Eq. (14). By taking Fourier transform of Eq. (22) one obtains the following expression for the mutual coherence function :

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2;\tau) = \sqrt{C_{12}}\exp[i\psi(\mathbf{r}_1, \mathbf{r}_2;\tau_0)]\Gamma(\mathbf{r}_1, \mathbf{r}_1;\tau - \tau_0). \tag{23}$$

On using Eqs. (2) and (23) and setting $\tau = \tau_0$, it follows at once that for such a field $|\gamma(\mathbf{r}_1, \mathbf{r}_2;\tau_0)| = 1$. Thus we have shown that if the light, at the two pinholes, is cross-spectrally pure, the converse of the theorem, discussed earlier, holds.

The main results which we have established in this Letter may be summarized by the following statements: (i) If an optical field is spatially completely coherent at a pair of points for a value τ_0 of the parameter τ, in the space-time formulation, then in the space-frequency formulation, it is spectrally completely coherent at that pair of points, *at every* frequency present in its spectrum. (ii) Even if every frequency component present in an optical field is spectrally completely coherent at a pair of points, the field itself may *not* be completely coherent in the space-time formulation for any value τ. However, if the light is cross-spectrally pure at that pair of points, it will be spatially coherent for some value $\tau = \tau_0$ in the space-time formulation.

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[1] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[2] L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965).
[3] H. Haken, *Laser Theory*, Encyclopedia of Physics Vol. XXV/2C (Springer-Verlag, Berlin, Heidelberg, New York, 1970).
[4] E. Wolf, Phys. Rev. Lett. **103**, 075501 (2009).
[5] E. Wolf, Phys. Lett. A **374**, 491 (2010).
[6] Z. Wang and G. Popescu, Appl. Phys. Lett. **96**, 051117 (2010).
[7] L. Mandel, J. Opt. Soc. Am. **51**, 1342 (1961).
[8] L. Mandel and E. Wolf, J. Opt. Soc. Am. **66**, 529 (1976).
[9] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).
[10] F. Zernike, Physica (Amsterdam) **5**, 785 (1938).
[11] E. Wolf, Proc. R. Soc. A **230**, 246 (1955).
[12] E. Wolf, *Introduction to the Theory of Coherence and Polarization of Light* (Cambridge University Press, Cambridge, 2007).
[13] S. Ponomarenko, H. Roychowdhury, and E. Wolf, Phys. Lett. A **345**, 10 (2005).
[14] H. Roychowdhury and E. Wolf, Opt. Commun. **248**, 327 (2005).
[15] E. Wolf "Statistical Similarity as a Unifying Concept of the Theories of Coherence and Polarization of Light", Opt. Commun. (to be published).
[16] E. Wolf, Nature (London) **326**, 363 (1987).
[17] D. F. V. James, J. Opt. Soc. Am. A **11**, 1641 (1994).
[18] M. Lahiri, E. Wolf, D. G. Fischer, and T. Shirai, Phys. Rev. Lett. **102**, 123901 (2009).
[19] An early work on this subject is described in A. T. Friberg and E. Wolf, Opt. Lett. **20**, 623 (1995). This issue was also partly addressed for fields of a special kind, namely propagation invariant fields, in J. Turunen, Opt. Express **16**, 20 283 (2008).
[20] Such a situation can be closely approximated by placing narrow-band filters in front of the pinholes.
[21] Narrow-band light may differ from monochromatic light, as regards its coherence proeprties. More specifically, a narrow-band light may not produce interference fringes of unit visibility in Young's two-pinhole experiment. This situation has been analyzed theoretically in E. Wolf, Opt. Lett. **8**, 250 (1983) and in M. Lahiri and E. Wolf, Phys. Lett. A **374**, 997 (2010). It has been verified experimentally by L. Basano, P. Ottonello, G. Rottigni, and M. Vicari, Appl. Opt. **42**, 6239 (2003) and by B. Kanseri and H. Kandpal, Opt. Lett. **35**, 70 (2010).