Perturbative Quantum Gravity as a Double Copy of Gauge Theory

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In a previous paper we observed that (classical) tree-level gauge-theory amplitudes can be rearranged to display a duality between color and kinematics. Once this is imposed, gravity amplitudes are obtained using two copies of gauge-theory diagram numerators. Here we conjecture that this duality persists to all quantum loop orders and can thus be used to obtain multiloop gravity amplitudes easily from gauge-theory ones. As a nontrivial test, we show that the three-loop four-point amplitude of $\mathcal{N}=4$ super-Yang-Mills theory can be arranged into a form satisfying the duality, and by taking double copies of the diagram numerators we obtain the corresponding amplitude of $\mathcal{N}=8$ supergravity. We also remark on a nonsupersymmetric two-loop test based on pure Yang-Mills theory resulting in gravity coupled to an antisymmetric tensor and dilaton.

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Although gauge and gravity theories have rather different physical behaviors we know that they are intimately linked. The celebrated AdS-CFT correspondence [1] is the most striking such example, linking maximally supersymmetric gauge theory to supergravity in AdS space. We also know that at weak coupling the tree-level (classical) scattering amplitudes of gauge and gravity theories are deeply intertwined because of the Kawai, Lewellen, and Tye (KLT) relations [2].

Recent years have seen a renaissance in the study of scattering amplitudes driven in part by the resurgence of collider physics with the recent start up of the Large Hadron Collider at CERN and by the realization that scattering amplitudes have far simpler and richer structures than visible from Feynman diagrams. Striking examples are the discoveries of twistor-space [3] and Grassmannian structures [4] in four dimensions for $\mathcal{N}=4$ super-Yang-Mills (sYM) theory, as well as interpolations between weak and strong coupling [5–7]. In another development we noted [8] that at tree level we could impose a duality between color and kinematics for gauge theories, without altering the amplitudes. This has important consequences in clarifying the tree-level relation between gravity and gauge theory. As we shall argue, this duality also greatly clarifies the multiloop structure of (super)gravity theories.

The key tool for our studies of loop amplitudes has been the unitarity method [9]. An important refinement which simplifies multiloop studies is the method of maximal cuts [10,11], which relies on generalized unitarity [12]. Here we will make use of these tools to present an all-loop extension of recently discovered tree-level relations. As we shall explain, this allows us to immediately write down multiloop gravity amplitudes directly from gauge-theory multiloop amplitudes once they have been organized to respect the duality between kinematics and color.

To understand the relationship between tree-level gravity and gauge-theory amplitudes, consider a gauge-theory amplitude where all particles are in the adjoint color representation. By exercising the trivial ability to absorb any higher-vertex terms into diagrams with only cubic vertices using factors of inverse propagators, we can choose to write it in the form,

$$\frac{1}{g^{n-2}}\mathcal{A}_n^{\mathrm{tree}}(1,2,3,\ldots,n)=\sum_i\frac{n_i c_i}{\prod_{\alpha_i}p_{\alpha_i}^2},\qquad(1)$$

where the sum runs over the $(2n-5)!!$ cubic diagrams and the product runs over all propagators (internal lines) $1/p_{\alpha_i}^2$ of each diagram. The c_i are the color factors obtained by dressing every three vertex with an $\tilde{f}^{abc}=i\sqrt{2}f^{abc}$ structure constant, and the n_i are kinematic numerator factors.

In general the n_i may be deformed under any shifts, $n_i\to n_i+\Delta_i$, where the Δ_i are arbitrary functions satisfying the constraint,

$$\sum_i\frac{\Delta_i c_i}{\prod_{\alpha_i}p_{\alpha_i}^2}=0.\qquad(2)$$

We call this a generalized gauge transformation, as some of the invariance does not correspond to a gauge transformation in the traditional sense.

The duality conjectured in Ref. [8] requires there to exist such a transformation from any valid representation to one where the numerators satisfy equations in one-to-one correspondence with the Jacobi identity of the color factors,

$$c_i=c_j-c_k\Rightarrow n_i=n_j-n_k.\qquad(3)$$

This duality is conjectured to hold to all multiplicity at tree level in a large variety of theories, including supersymmetric extensions of Yang-Mills theory. Surprisingly, this duality implies new nontrivial relations between the color-ordered partial amplitudes of gauge theory [8]. A proof of these relations has been made using monodromy for integrations in string theory [13].

Perhaps more striking is the observation [8] that once the gauge-theory amplitudes are arranged into a form satisfying the duality (3), gravity tree amplitudes are given by,

$$-i\left(\frac{2}{\kappa}\right)^{(n-2)}\mathcal{M}_n^{\rm tree}(1,2,\ldots,n)=\sum_i\frac{n_i\tilde{n}_i}{\prod_{\alpha_i}p_{\alpha_i}^2},\qquad(4)$$

where the $\tilde{n}$ represent numerator factors of a second gauge theory, the sum runs over the same diagrams as in Eq. (1), and κ is the gravitational coupling constant. This form of gravity tree amplitudes has been verified explicitly in field theory through eight points [8] using the KLT relations.

These properties are now being understood in string theory [14–16]. The heterotic string, in particular, offers keen insight into these properties because of the parallel treatment of color and kinematics [15]. A field theory proof of Eq. (4) has now been given [17] for an arbitrary number of external legs, assuming the duality (3) holds. We note that the invariance (2) implies that only one family of numerators (n or $\tilde{n}$) needs to satisfy the duality (3), a consequence independently realized by Kiermaier—see Ref. [17] for details. Below we will confirm this property for the $\mathcal{N}=8$ supergravity three-loop four-point amplitude.

If both families of kinematic factors are for $\mathcal{N}=4$ sYM theories, the gravity theory amplitudes are for $\mathcal{N}=8$ supergravity (sugra). If pure-Yang-Mills theory is instead used, the obtained gravity amplitudes correspond to Einstein gravity coupled to an antisymmetric tensor and dilaton; the n-graviton tree-level amplitudes of this theory correspond to pure gravity. Additionally, the tilde numerator factors ($\tilde{n}$) need not come from the same theory as the untilde factors. This allows for the construction of gravity amplitudes with varying amounts of supersymmetry.

In this Letter we conjecture that diagram numerators satisfying the duality (3) can be found at loop level as well whenever the tree amplitudes have this property. As such, gauge theory and gravity amplitudes in these theories would be related via,

$$\frac{(-i)^L}{g^{n-2+2L}}\mathcal{A}_n^{\rm loop}=\sum_j\int\prod_{l=1}^{L}\frac{d^Dp_l}{(2\pi)^D}\frac{1}{S_j}\frac{n_jc_j}{\prod_{\alpha_j}p_{\alpha_j}^2},\qquad(5)$$

$$\frac{(-i)^{L+1}}{(\kappa/2)^{n-2+2L}}\mathcal{M}_n^{\rm loop}=\sum_j\int\prod_{l=1}^{L}\frac{d^Dp_l}{(2\pi)^D}\frac{1}{S_j}\frac{n_j\tilde{n}_j}{\prod_{\alpha_j}p_{\alpha_j}^2},\qquad(6)$$

where the sums now run over all distinct n-point L-loop diagrams with cubic vertices. These include distinct permutations of external legs, and the S_j are the (internal) symmetry factors of each diagram. As at tree level, at least one family of numerators (n_j or $\tilde{n}_j$) for gravity must be constrained to satisfy the duality (3). (For pure gravity, extra projectors are needed to obtain loop-level amplitudes from the direct product of two pure Yang-Mills theories.)

Our loop-level conjecture is largely motivated by the unitarity analysis along the lines presented in Ref. [8], decomposing loop amplitudes into tree amplitudes whose duality properties have been confirmed in multiple studies [8,13–17], as well as by the very recent construction of relevant Lagrangians whose diagrams satisfy the duality [17]. Note, also, that it is straightforward to check that the known one and two-loop four-point amplitudes of $\mathcal{N}=4$ sYM theory and $\mathcal{N}=8$ sugra, as given in ref. [18], satisfy the conjecture.

The key aspect of our conjecture is that gauge-theory multiloop amplitudes admit an organization of the integral numerators making manifest the duality with color (3). As a consequence, the gravity loop amplitudes of Eq. (6) follow from applying the unitarity method and the tree-level formula (4).

To test our conjecture in a rather nontrivial case, we consider the three-loop four-point amplitude of $\mathcal{N}=8$ sugra. This amplitude has already been studied in some detail in Refs. [11,19]. Our task is to see if we can organize the four-point three-loop amplitude of $\mathcal{N}=4$ sYM so its numerator factors satisfy the duality (3) with all internal momenta off shell, and then to check if the expression constructed via squaring those numerator factors is the four-point three-loop amplitude of $\mathcal{N}=8$ supergravity.

We identify the set of diagrams with cubic vertices whose color factors mix via the color-Jacobi identity to the nine diagram topologies used in constructing the four-point three-loop $\mathcal{N}=4$ sYM amplitude [11]. This gives a total of 25 distinct three-loop diagrams to consider, up to relabelings. Any contact terms will be included as inverse propagators in the numerators.

We start by dressing each of the 25 distinct Feynman integrands with generic numerator polynomials containing a set of arbitrary parameters, which will be fixed by various constraints. We include only those Lorentz products not simply related to the others via momentum conservation. After factoring out a universal factor of the color-ordered tree amplitude and Mandelstam invariants $stA_4^{\rm tree}(1,2,3,4)$, which appears in each term for $\mathcal{N}=4$ sYM, the remaining polynomial has total degree four in the external and loop momenta. In order to respect the known power counting, we require that the numerator of each diagram is at most quadratic in the loop momenta. We also require that each kinematic numerator respect the symmetries of the diagram, accounting for the antisymmetry of each cubic vertex under an interchange of any two legs.

To initially constrain the parameters, we use the unitarity method to compare each cut of the ansatz against the corresponding cut of the $\mathcal{N}=4$ sYM amplitude,

$$\sum_{\rm states}A_{(1)}^{\rm tree}A_{(2)}^{\rm tree}A_{(3)}^{\rm tree}\cdots A_{(m)}^{\rm tree},\qquad(7)$$

invoking kinematics that place all cut lines on shell, $l_i^2=0$. Once a solution consistent with a complete set of cuts is found, we have the amplitude. From Ref. [11] we know that for this amplitude the maximal and near maximal cuts are sufficient (although we also evaluated other complete sets of cuts as a cross check). We perform all cut evaluations in D dimensions using the known D-dimensional results [19] for the cuts. Matching to the cut conditions determines the amplitude but still allows freedom, as contact terms can be assigned to various diagrams.

To impose the duality (3) on the amplitude, we step through every propagator in each diagram, ensuring that all duality relations hold off shell. On any diagram, we can describe any internal line, carrying some momentum l_s, in terms of formal graph vertices $V(p_a, p_b, l_s)$, and $V(-l_s, p_c, p_d)$ where the p_i are the momenta of the other legs attached to l_s, as illustrated on the left side of Fig. 1. The duality (3) requires the following:

$$\begin{aligned} n(\{V(p_a, p_b, l_s), V(-l_s, p_c, p_d), \ldots\}) \\ = n(\{V(p_d, p_a, l_t), V(-l_t, p_b, p_c), \ldots\}) \\ + n(\{V(p_a, p_c, l_u), V(-l_u, p_b, p_d), \ldots\}), \end{aligned} \qquad (8)$$

where n represents the numerator associated with the diagram specified by the set of vertices, the omitted vertices are identical in all three diagrams, and $l_s \equiv (p_c + p_d)$, $l_t \equiv (p_b + p_c)$ and $l_u \equiv (p_b + p_d)$ in the numerator expressions. There is one such equation for every propagator in every diagram. Solving the system of distinct equations enforces the duality conditions (3).

Imposing the duality on the ansatz, at this point, completely fixes the form of the amplitude. We find that only the 12 diagrams shown in Fig. 2 contribute, with the numerator factors given in Table I. As noted above, a direct consequence of unitarity and the tree-level duality is that squaring these numerator factors should give the numerators for $\mathcal{N} = 8$ sugra. We verified this is indeed the case using a complete set of cuts of the known result [11,19]. Interestingly, by cutting one or two internal legs of the three-loop four-point gauge-theory amplitude, we obtain eight-point one-loop and six-point two-loop amplitudes also satisfying the duality (3) off shell, albeit with sums over states and restricted kinematics.

We also construct another version of the three-loop four-point $\mathcal{N} = 8$ sugra expression via (6) using the n_i given in Table I of the present Letter and the correct, but duality violating , $\tilde{n}_i$ from table I of Ref. [11]. We find that this is also a valid representation of the $\mathcal{N} = 8$ sugra three-loop four-point amplitude, providing a strong consistency check

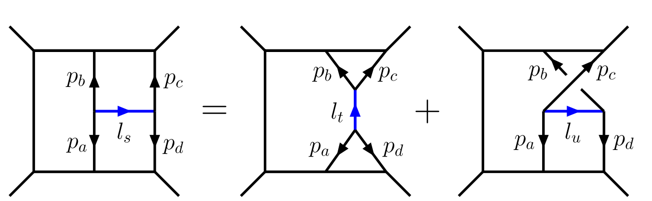

FIG. 1 (color online). The loop-level numerator identity enforced by the duality (3) on propagator l_s of the leftmost diagram equates that diagram's numerator with the sum of the numerators of the rightmost diagrams.

on Table I and our conjecture. Such representations are valid at loop level by the same argument as at tree level: there exists a generalized gauge transformation (2) transforming it to one where both n_i and $\tilde{n}_i$ satisfy the duality.

An important feature of the supergravity solution displayed in Table I is that each contribution to Eq. (6) has no worse power counting than the leading behavior of the $\mathcal{N} = 4$ sYM amplitude. This is worthy of further study, especially as relevant to four loops and beyond [20].

Perhaps the most surprising feature of our construction is that, with the duality (3) imposed, the only cut information actually required to construct the complete $\mathcal{N} = 4$ sYM amplitude is that under maximal cut conditions, the numerator of diagram (e) is $s\tau_{45}$. This suggests that the constraints of this duality are powerful enough so that only a relatively small subset of unitarity cuts is necessary to fully determine higher-loop amplitudes.

The above three-loop example has maximal supersymmetry. Naturally there is a question of whether our loop-level conjecture (6) relies on supersymmetry. To see that it does not we need only look at the two-loop four-point identical-helicity amplitude in pure Yang-Mills given in Ref. [21]. As noted in Ref. [8] the duality is manifest in this example when cut conditions are imposed. This property also persists with off shell loop momenta, and when the numerators are squared, we obtain the correct identical-helicity four-graviton amplitude in the theory of gravity coupled to an antisymmetric tensor and dilaton.

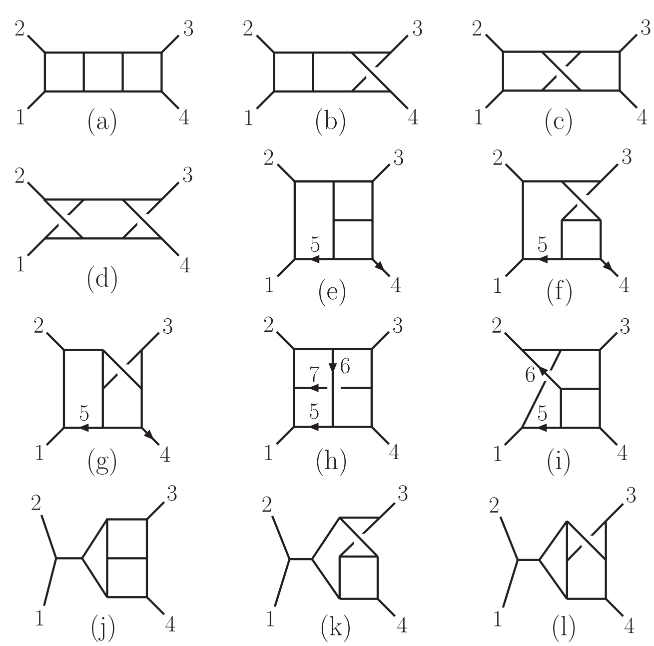

FIG. 2. Loop diagrams contributing to both $\mathcal{N} = 4$ sYM and $\mathcal{N} = 8$ sugra three-loop four-point amplitudes. Integrals (6) are specified by combining their propagators with numerator factors given in Table I. The (internal) symmetry factor for diagram (d) is $S_{(\mathrm{d})} = 2$, the rest are unity. All distinct external permutations of each diagram contribute.

TABLE I. The numerator factors of the integrals $I^{(x)}$ in Fig. 2. The first column labels the integral, the second column the relative numerator factor for $\mathcal{N}=4$ super-Yang-Mills theory. The square of this is the relative numerator factor for $\mathcal{N}=8$ supergravity. An overall factor of stA_4^{tree} has been removed, s, t, u are Mandelstam invariants corresponding to $(k_1+k_2)^2$, $(k_2+k_3)^2$, $(k_1+k_3)^2$ and $\tau_{ij}=2k_i\cdot l_j$, where k_i and l_j are momenta as labeled in Fig. 2.

Integral $I^{(x)}$	$\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator
(a)–(d)	s^2
(e)–(g)	$[s(-\tau_{35}+\tau_{45}+t)-t(\tau_{25}+\tau_{45})+u(\tau_{25}+\tau_{35})-s^2]/3$
(h)	$[s(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)+t(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2]/3$
(i)	$[s(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)+t(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\tau_{25}+s^2]/3$
(j)–(l)	$s(t-u)/3$

In summary, we propose that the gauge-theory duality between color and kinematic numerators imposed in Ref. [8] carries over naturally to loop level. This allows the expression of numerators of gravity diagrams using two copies of gauge-theory ones. To test this idea, we discussed two nontrivial examples, one in some detail. The known connection between scattering amplitudes of $\mathcal{N}=4$ super-Yang-Mills theory at weak [5] and strong coupling [7], suggests that the duality between color and kinematics will also impose nontrivial constraints at strong coupling. It also seems likely that an analogous duality should hold in higher-genus perturbative string theory. It has not escaped our attention that should the duality between color and kinematics hold to all loop orders it would have important implications in studies of the ultraviolet behavior of quantum gravity theories (for recent reviews see Refs. [22]). We close by remarking that the double-copy gravity numerators hint at some notion of compositeness, albeit with a rather novel structure. This structure may very well have important consequences outside of perturbation theory.

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