Sub-Fourier Characteristics of a δ-kicked-rotor Resonance

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We experimentally investigate the sub-Fourier behavior of a δ-kicked-rotor resonance by performing a measurement of the fidelity or overlap of a Bose-Einstein condensate exposed to a periodically pulsed standing wave. The temporal width of the fidelity resonance peak centered at the Talbot time and zero initial momentum exhibits an inverse cube pulse number ($1/N^3$)-dependent scaling compared to a $1/N^2$ dependence for the mean energy width at the same resonance. A theoretical analysis shows that for an accelerating potential the width of the resonance in acceleration space depends on $1/N^3$, a property which we also verify experimentally. Such a sub-Fourier effect could be useful for high precision gravity measurements.

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The quantum δ-kicked-rotor (QDKR) has proved to be an excellent testing ground for theoretical and experimental studies of chaos in the quantum domain [1]. An experimental version of this system in the form of the kicked particle is achieved by exposing cold atoms to N pulses of an off-resonant standing wave of light [1,2]. Ever since its realization, the QDKR has revealed a rich variety of effects including dynamical localization [3], quantum accelerator modes [4,5], quantum ratchets [6,7], and quantum resonances [2,8,9]. Such resonances appear for pulses separated by rational fractions of a characteristic time called the Talbot time and can be observed as sharp peaks in the mean energy of the system [10]. The width of these peaks has been found to scale as $1/N^2$, a sub-Fourier effect attributed to the nonlinear nature of the QDKR and explained using a semiclassical picture [11]. Away from the resonances, dynamical localization sets in, characterized by the quantum suppression of classical momentum diffusion beyond a "quantum break time" [3]. This property, unique to quantum dynamics in the chaotic regime, was utilized to discriminate between two driving frequencies of the QDKR with sub-Fourier resolution [12].

High-precision measurements using quantum-mechanical principles have been carried with atom interferometers for many years [13]. Such devices were used to determine the Earth's gravitational acceleration [14], fine structure constant α [15], and the Newtonian constant of gravity [16]. The promise of the QDKR as a candidate for making these challenging measurements has begun to be realized [17]. Recently a scheme was proposed for measuring the overlap or fidelity between a near-resonant δ-kicked-rotor state and a resonant state via application of a tailored pulse at the end of a rotor pulse sequence [18]. It predicted a $1/N^3$ scaling of the temporal width of the fidelity peak. In this paper we report on the observation of such fidelity resonance peaks and their sub-Fourier nature. Figure 1 illustrates a plot of the fidelity (fraction of atoms in the zeroth order momentum state) vs pulse period obtained by the application of an overlap pulse at the end of five rotor kicks. For comparison, we also plot the mean energy of the rotor sequence. It can be seen that even for relatively few kicks the fidelity peak is significantly narrower. We also investigated the sensitivity of this fidelity resonance to an accelerating rotor. As will be seen, our calculations indicate that the width of the fidelity peak vs acceleration decreases at a sub-Fourier rate of $1/N^3$. We confirm this result with experiments.

The dynamics of a periodically kicked atom in the presence of a linear potential is described by the quantum δ-kicked accelerator (QDKA) Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2} + \frac{\eta}{\tau}\hat{X} + \phi_d \cos(\hat{X}) \sum_{t=1}^{N} \delta(t' - t\tau). \qquad (1)$$

$\hat{P}$ is the momentum (in units of two photon recoils, $\hbar G$) that an atom of mass M acquires from short, periodic

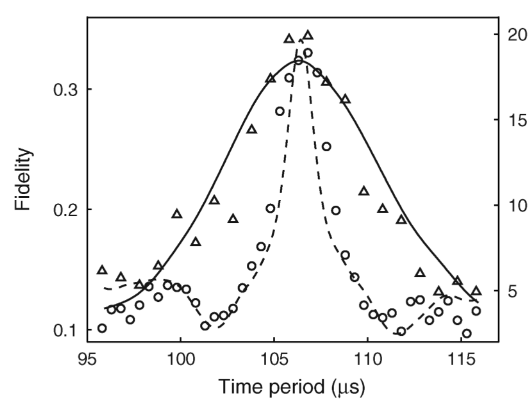

FIG. 1. Experimentally measured fidelity distribution (circles) due to five kicks of strength $\phi_d = 0.8$ followed by a π phase shifted kick of strength $5\phi_d$. The mean energy (triangles) of the same five kick rotor is shown for comparison. Numerical simulations of the experiment for a condensate with momentum width $0.06\hbar G$ are also plotted for fidelity (dashed line) and mean energy (solid line). The amplitude and offset of the simulated fidelity were adjusted to account for the experimentally imperfect reversal phase.

pulses of a standing wave with a grating vector $G = 2\pi/\lambda_G$ (λ_G is the wavelength of the standing wave). $\hat{X}$ is position in units of G^{-1} and $\eta = Mg'T/\hbar G$, g' being its acceleration between pulses separated by T, the pulse period. $\phi_d = \Omega^2 \Delta t/8\delta_L$ represents the kicking strength of a pulse of length Δt, Ω is the Rabi frequency, and δ_L the detuning of the kicking laser from the atomic transition. t' is the continuous time variable and $\tau = 2\pi T/T_{1/2}$ is the scaled pulse period.

For the case $\eta = 0$, Eq. (1) reduces to the usual QDKR Hamiltonian [2]. For now we restrict ourselves to this situation. Primary quantum resonances are seen for pulses separated by integer multiples of the half-Talbot time, $T_{1/2} = 2\pi M/\hbar G^2$ or $\tau = 2\pi$. Adjacent momentum orders evolve phases which are integer multiples of 2π during this time period resulting in a quadratic growth in the rotor mean energy, $\langle E \rangle = 2E_r \phi_d^2 N^2$, where $E_r = \hbar^2 G^2/8M$ is the photon recoil energy. The width of the mean energy around the resonance time was found to decrease as $1/(N^2 \phi_d)$ [9,11].

In order to demonstrate the role of the relative phase deviations of the contributing momentum states near such a resonance, a "fidelity" test for the QDKR was proposed in Ref. [18]. In this scheme, a kick changed in phase by π and carrying a strength of $N\phi_d$ is applied at the end of the N rotor kicks. The fidelity is then defined as $F = |\langle \beta | U_r U^N | \beta \rangle|^2$ where $U = \exp(-i\frac{\tau}{2}\hat{P}^2) \times \exp[-i\phi_d \cos(\hat{X})]$ describes the one period evolution, $U_r = \exp[iN\phi_d \cos(\hat{X})]$ is the overlap pulse, and β is the fractional part of the momentum. F therefore gives the probability of the revival of the initial state and is measured by the fraction of atoms which have returned to the initial zero momentum state. Using a perturbative treatment, it was shown that near the resonance at the Talbot time, $\tau = 4\pi$, the fidelity is [18]

$$F(\epsilon, \beta = 0) \simeq J_0^2\left(\frac{1}{12} N^3 \phi_d^2 \epsilon\right), \qquad (2)$$

where $\epsilon = \tau - 4\pi$. The width of such a peak in ϵ therefore changes to $1/(N^3 \phi_d^2)$, displaying a stronger sub-Fourier dependence on the number of kicks than the mean energy.

Our experiment is performed by producing a Bose-Einstein condensate of 20 000 Rb87 atoms in the $5S_{1/2}$, $F = 1$, $m_F = 0$ level in an optical trap [5,19]. After being released from the trap, the condensate is exposed to a horizontal standing wave created by two beams of wavelength $\lambda = 780$ nm light detuned 6.8 GHz to the red of the atomic transition. The wave vector of each beam was aligned $\theta = 52°$ to the vertical. This created a horizontal standing wave with a wavelength of $\lambda_G = \lambda/2\sin\theta$ and a corresponding Talbot time of 106.5 μs. Two acousto-optic modulators controlled the pulse lengths as well as the relative frequencies of the kicking beams enabling the control of the acceleration and initial momentum of the standing wave with respect to the condensate. The kicking pulse length was $\Delta t = 0.8$ μs with a $\phi_d \approx 0.6$. For the last kick the phase of one of the acousto-optic modulators rf driving signal was changed by π which shifted the standing wave by half a wavelength. In order to keep this final overlap pulse within the Raman-Nath regime we varied the intensity rather than the pulse length to create a kick strength of $N\phi_d$. This was done by adjusting the amplitudes of the rf waveforms driving the kicking pulse. Dephasing primarily due to vibrations made the reversal process inconsistent for $N > 6$ [20]. To reduce this, the standing wave at each kick was shifted by half a wavelength with respect to the previous kick. That is, the summation in Eq. (1) becomes $\sum_{t=1}^{N}(-1)^{t-1}\delta(t' - t\tau)$. This had the effect of shifting the Talbot time resonance to $T_{1/2}$. Consequently the reduced experimental time led to much improved results. Following the entire kicking sequence we waited 8 ms for the different momentum orders to separate before the atoms were absorption imaged.

From the time-of-flight images fidelity F is measured as the fraction of atoms which have reverted back to the zeroth order momentum state, that is $F = P_0/\sum_n P_n$ where P_n is the number of atoms in the nth momentum order. To facilitate the analysis of the data, all of the resonance widths ($\delta\epsilon$) were scaled to a reference kick number of $N = 4$. That is, we define a scaled fidelity width $\Delta\epsilon = \delta\epsilon/\delta\epsilon_{N=4}$ for each scaled kick number $N_s = N/4$ and recover $\log\Delta\epsilon = -3\log N_s$ using Eq. (2). For each kick, a scan is performed around the resonance time. To ensure the best possible fit of the central peak of the fidelity spectrum to a Gaussian, the time is scanned between values which make the argument of J_0^2 of Eq. (2) ≈ 2.4 so that the first side lobes are only just beginning to appear. Figure 2(a) plots the logarithm of the FWHM for four to nine kicks scaled to the fourth kick. A linear fit to the data gives a slope of -2.73 ± 0.19 giving a reasonable agreement with the predicted value of -3 within the experimental error. As seen in the same figure, the results are close to the numerical simulations which take into account the finite width of the initial state of $0.06\hbar G$ [7]. We also compared the resonance widths of the kicked-rotor mean energy $\langle E \rangle$ to that of the fidelity widths. As in the fidelity, the plotted values $\Delta\langle E \rangle$ have been normalized to that of the fourth kick. On the log scale, the width of each peak gets narrower with the kick number with a slope of -1.93 ± 0.21 [Fig. 2(a)] in agreement with previous results [9,21]. As a further test of Eq. (2), the variation in the widths of the fidelity and mean energy peaks were studied as a function of ϕ_d. Figure 2(b) shows the fidelity width changing with a slope of -1.96 ± 0.3, close to the predicted value of -2. This again scales faster than the mean energy width which decreases with a slope of -0.88 ± 0.24 (the theoretical value being -1).

The resonances studied here appear for pulses separated by the Talbot time and an initial momentum state of $\beta = 0$.

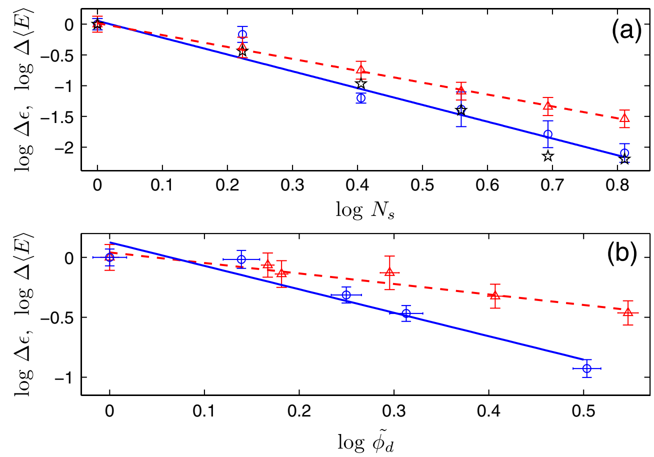

FIG. 2 (color online). Experimentally measured fidelity (circles) and mean energy (triangles) widths (FWHM) as a function of (a) the number of pulses, and (b) the kicking strength $\tilde{\phi}_d$ scaled to ϕ_d of the first data point. In (a), the data are for four to nine kicks in units normalized to the fourth kick. Error bars in (a) are over three sets of experiments and in (b) 1σ of a Gaussian fit to the distributions. Dashed lines are linear fits to the data. Stars are numerical simulations for an initial state with a momentum width of $0.06\hbar G$.

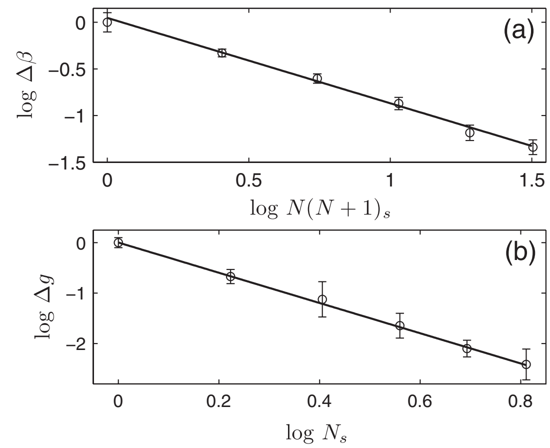

FIG. 3. (a) Variation of the fidelity peak width around $\beta = 0$ as a function of kick number $N(N+1)_s = N(N+1)/20$ scaled to the fourth kick. The straight line is a linear fit to the data with a slope of -0.92 ± 0.06. Error bars as in Fig. 2(b). (b) Dependence of the acceleration resonance peak width on N in units scaled to the fourth kick. Error bars are over three sets of experiments.

Away from this resonant β, phase changes in the amplitudes of the different momentum orders lead to a fidelity which depends on the initial momentum as $F(\epsilon = 0, \beta) = J_0^2[2\pi\phi_d N(N+1)\beta]$ [18]. The peak width in β space is thus expected to change as $1/[N(N+1)]$ around $\beta = 0$, as against a $1/N$ scaling of the mean energy width [9]. To verify this, the initial momentum of the condensate with respect to the standing wave was varied and the kicking sequence applied. The experimentally measured widths $\Delta\beta = \delta\beta/\delta\beta_{N=4}$ in Fig. 3(a) display a scaling of $\Delta\beta \propto [N(N+1)]^{-0.92}$ close to the theoretical value.

For an initial state $|\beta + n\rangle$, the wave function acquires a nonzero phase during the free evolution even at the Talbot time. Therefore the final kick performs a velocity selective reversal, preferentially bringing back atoms closer to an initial momentum of $\beta = 0$. This is similar to the time-reversed Loschmidt cooling process proposed in Refs. [22,23], although in that technique a forward and reverse path situated on either side of the resonant time was used in order to benefit from the chaotic dynamics. To observe this effect, the current scheme offers an experimental advantage in terms of stability due to the reduced length of the pulse sequence. Here, only a single pulse performs the velocity selection at the end, whereas in the Loschmidt technique N phase reversed kicks separated by a finite time are used. Figure 4 demonstrates the reduction of the momentum distribution width. Accompanying this decrease is a drop in the peak height. Our simulations and the results of Ref. [23] predict that for the case of a noninteracting condensate, this should remain constant. In addition to interactions, we expect experimental imperfections in the fidelity sequence to play a role in the smaller peak densities with increasing kick numbers. We performed the same experiment 4.5 ms after the Bose-Einstein condensate was released from the trap when the mean field energy had mostly been transformed to kinetic energy in the expanding condensate. A similar reduction in the momentum width of the reversed state, along with a decrease in the peak density, was observed.

We now investigate the behavior of fidelity in the presence of acceleration, i.e. $\eta \neq 0$. The state of the QDKA Eq. (1) after N kicks is $|\psi(N\tau)\rangle = \sum_n c_n|n+\beta\rangle$ where n is the integer part of momentum $\hat{P}$. The expansion coefficients c_n are $c_n(\epsilon, \beta, \eta) = \langle n+\beta|\hat{U}_{g_N}\cdots\hat{U}_{g_2}\hat{U}_{g_1}|\beta\rangle$. $\hat{U}_{g_t} = \exp[-i\frac{\tau}{2}(\hat{p} + t\eta + \frac{\eta}{2})^2]\exp[-i\phi_d\cos(\hat{X})]$ is the tth kick evolution operator in the freely falling frame obtained after a gauge transformation of the Hamiltonian (1) which restores the conservation of *quasimomentum* β [24]. Close to the resonances, we have $F(\epsilon, \beta, \eta) \simeq |\sum_n J_n^2(N\phi_d)\exp(-i\Theta_n)|^2$, where $\Theta_n = \frac{\partial\theta_n}{\partial\epsilon}|\epsilon + \frac{\partial\theta_n}{\partial\beta}|\beta + \frac{\partial\theta_n}{\partial\eta}|\eta$ describes the effect of deviations from resonance on the coefficients c_n. Using a procedure detailed in Ref. [18], one can show that $\frac{\partial\theta_n}{\partial\eta}|_{(\epsilon=\beta=\eta=0)} = \frac{\frac{\partial c_n}{\partial\eta}|}{ic_n(0,0,0)} = -4\pi n N^2/3$, where we have kept terms in N^2. Finally we arrive at the fidelity in the presence of acceleration,

$$F(\eta, \epsilon = \beta = 0) = J_0^2\left(\frac{4\pi}{3}N^3\phi_d\eta\right). \tag{3}$$

Thus the width of such a peak centered at the resonant zero acceleration should drop as $1/N^3$. In order to verify the above result, the standing wave was accelerated during the application of the pulses. This acceleration was scanned

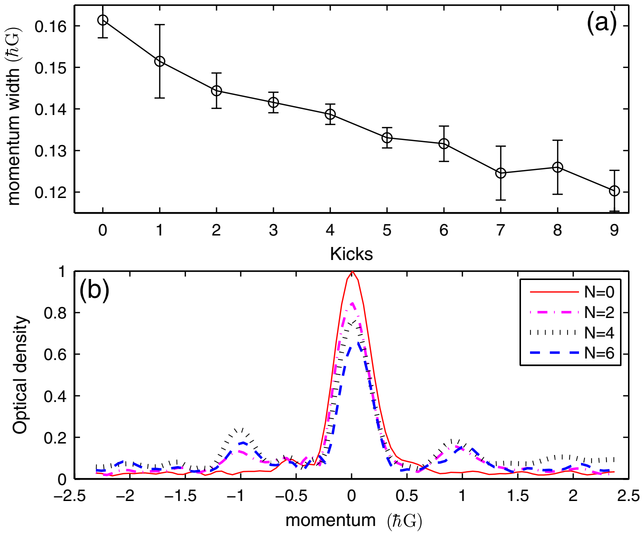

FIG. 4 (color online). (a) Momentum width of the reversed zeroth order state as a function of kick number. Error bars are an average over three experiments. (b) Optical density plots for the initial state [(red) solid curve] and kick numbers 2 [(magenta) dot-dashed curve], 4 [(black) dotted curve], and 6 [(blue) dashed curve] after summation of the time-of-flight image along the axis perpendicular to the standing wave.

across the resonant zero value and readings of the fidelity collected. Since a typical value of the half width at half maximum is $\eta = 0.05$ for $N = 4$ (corresponding to an acceleration of 4 $\mathrm{m/s^2}$), the perturbative treatment of acceleration on fidelity used above is justified. Figure 3(b) plots the experimental data for four to nine kicks, where the widths of the peaks decrease with a slope of -3.00 ± 0.23 in excellent agreement with the theory.

In conclusion, we performed experimental measurements of the fidelity widths of a δ-kicked-rotor state near a quantum resonance. The width of these peaks centered at the Talbot time decreased at a rate of $N^{-2.73}$ comparable to the predicted exponent of -3. By comparison, the mean energy widths were found to reduce only as $N^{-1.93}$. Furthermore, the fidelity peaks in momentum space changed as $[N(N+1)]^{-0.92}$, also consistent with theory. The reversal process used in the fidelity experiments led to a decrease in the momentum distribution of the final zeroth order state by $\sim$25% (for $N = 9$) from the initial width. The sub-Fourier dependencies of the mean energy and fidelity observed here are characteristic of the dynamical quantum system that is the QDKR [18]. The narrower resonances of the fidelity scheme could be exploited in locating the resonance frequency with a resolution below the limit imposed by the Fourier relation. This can help determine the photon recoil frequency ($\omega_r = E_r/\hbar$) which together with the photon wavelength enables measurement of the fine structure constant with a high degree of precision [15,17]. We also demonstrated a N^{-3} dependence of the resonance width in acceleration space in accordance with the extended theory. The sensitivity of an atom-interferometer-based gravimeter scales as the square of the loop time, hence the pursuit of large area interferometers to improve accuracy [14]. By comparison, the fidelity is responsive to the gravitational acceleration g with the cube of the ''time'' N, leading to the possibility of higher precision measurements. One could perform a fidelity measurement on a freely falling condensate exposed to kicks accelerating at the local value of g (to realize $\eta \ll 1$). Variation in g would then manifest itself as a shift of the resonant acceleration. A parts per billion precision [14] would require a judicious selection of the parameters (N, ϕ_d, T), for instance (150, 10, $16T_{1/2}$). Such a resolution, though not feasible in the current setup without addressing stability-related issues, could be possible with future refinements.

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