Supertransient Magnetohydrodynamic Turbulence in Keplerian Shear Flows

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A subcritical transition to turbulence in magnetized Keplerian shear flows is investigated by using a statistical approach. Three-dimensional numerical simulations of the shearing box equations with zero net magnetic flux are employed to determine the transition from decaying to sustained turbulence as a function of the magnetic Reynolds number R_m. The results reveal no clear transition to sustained turbulence as the average lifetime of the transients grows as an exponential function of R_m, in accordance with a type-II supertransient law.

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There has been much progress in recent years on the understanding of the nature of subcritical transitions to turbulence in hydrodynamic flows, wherein the laminar state is linearly stable for all Reynolds numbers and finite amplitude perturbations are necessary to drive the system towards a turbulent state. Numerical and experimental works indicate that turbulence in systems like plane-Couette and pipe flows is a transient phenomenon [1,2], although this is still a disputed topic [3,4]. Although much has been accomplished for hydrodynamic turbulence in this area, the same cannot be said about magnetohydrodynamic (MHD) turbulence in systems with subcriticality, for which statistical studies are scarce.

A classical problem of subcritical transition to turbulence is provided by accretion disks, where, in the absence of magnetic fields and when shear is the only stratification, Keplerian flows are linearly stable [5]. The rate of mass accretion in the disk depends on the outward transport of angular momentum, and one would think that turbulence is the natural candidate for transport. However, both numerical [6–8] and laboratory [9] experiments have revealed that, if a subcritical hydrodynamic transition to turbulence were to exist in Keplerian flows, then the associated turbulent transport of angular momentum is very weak and unable to match observational constraints.

When a mean vertical field is applied, the magnetorotational instability [10] provides a linear instability that may result in a strong MHD turbulence capable of generating adequate rates of angular momentum transport [11,12]. If there is no imposed background magnetic field, the original (seed) field can decay with time due to finite resistivity. In that case, some type of nonlinear dynamo process is necessary to amplify and sustain the seed field, and, consequently, the instability in zero net flux simulations is nonlinear [13]. Rincon, Ogilvie, and Proctor [14] identified a self-sustaining dynamo solution in a magnetized rotating plane-Couette flow in the Keplerian regime, but the steady

solutions were found only in a narrow range of low values of the Reynolds number. Other works have reported the existence of nonlinear dynamos in turbulent MHD regimes [13], yet little is known about how turbulence evolves as a function of the magnetic Prandtl number P_m (the ratio of viscosity to resistivity) and whether it can be sustained for long runs or not. Recently, Fromang *et al.* [15] studied the onset of turbulence in accretion disks by using an unstratified shearing box with zero net magnetic flux and found that turbulence disappears when the magnetic Prandtl number falls below a critical value that seems to be a decreasing function of the kinetic Reynolds number. Note, however, that vertical stratification might change this picture [16].

The present Letter explores the onset of turbulence in three-dimensional, zero net flux, MHD simulations of Keplerian shear flows as a function of the magnetic Reynolds number R_m. We focus on the statistical aspects of the subcritical transition, which is an approach that has been unexplored in MHD simulations, despite its popularity in the hydrodynamics community. From our results, there seems to be no clear transition to sustained turbulence. Instead, the average lifetime of the turbulence τ grows as an exponential function of R_m, following a type-II supertransient law [17].

We solve the shearing box equations [18], which represent the dynamics in a box moving with the disk's angular velocity $\Omega(r_0)$, where r_0 is a fiducial radius that determines the location of the center of the box. In Cartesian coordinates, with $\phi \rightarrow y$ and $r \rightarrow r_0 + x$, the shearing box equations read as

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{1}{\mu_0 \rho}(\nabla \times \mathbf{B}) \times \mathbf{B} - 2\mathbf{\Omega}$$
$$\times \mathbf{v} + 2\Omega S x \hat{\mathbf{x}} + \nu \nabla^2 \mathbf{v}, \quad (1)$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{v} = 0, \qquad (3)$$

where $\Omega = r^{-3/2}$ for a Keplerian disk, $S = -r\partial_r\Omega = (3/2)\Omega$, η is the magnetic diffusivity, μ_0 is the magnetic permeability, ρ is the gas density, and P is the pressure. The fluid velocity can be decomposed as $\mathbf{v} = \mathbf{v}_0 + \mathbf{u}$, where the steady state solution is given by the shear flow $\mathbf{v}_0 = -Sx\hat{\mathbf{y}}$ and $\mathbf{u}$ is the perturbation field. The equations for $\mathbf{u}$ are solved with the pseudospectral code described by Lesur and Longaretti [8,19] by using shearing sheet boundary conditions [11]. The xyz aspect ratio employed is $1 \times \pi \times 1$. The kinetic and magnetic Reynolds numbers are defined as $\text{Re} = Sd^2/\nu$ and $R_m = Sd^2/\eta$, respectively, where d is the shearwise size of the box. Time is measured in shear time units $1/S$.

Transition to turbulence is investigated by fixing Re while varying R_m. The value of Re is taken from Fig. 11 of Fromang *et al.* [15], where for $\text{Re} = 3125$ there is a transition to turbulence somewhere between $R_m = 6250$ and $R_m = 12\,500$, and, thus, the magnetic Prandtl number $P_m = R_m/\text{Re} > 1$. Since we are not interested in the kinematic dynamo regime, the initial growth of a small seed magnetic field is not important to our discussions. Our question is, given a turbulent state, can it be sustained for all time? If not, how long does it take, on average, to decay? The turbulent states are obtained from long runs where the velocity and magnetic fields are initialized with a large-scale random noise (only the four largest modes are nonzero at $t = 0$) with amplitude of the order of 1. After some time, energy spreads to other modes and the random fields settle to a turbulent state. Figure 1 illustrates the typical turbulent patterns found at $\text{Re} = 3125$ and $R_m = 11\,000$ for the radial component of the velocity (upper panel) and magnetic (lower panel) fields at $t = 500$. Since $P_m > 1$ and there is no proper large-scale dynamo, the characteristic spatial scales in the magnetic fields are smaller than in the velocity fields. The numerical resolution employed is $64 \times 128 \times 64$, which is the same resolution used for this problem in Ref. [15]. The convergence of the code is checked by comparing the time-averaged power spectra obtained with this run with a higher-resolution run ($128 \times 256 \times 128$). The results are shown in Fig. 2, where the solid black line represents the lower-resolution kinetic energy spectrum and the solid red line the lower-resolution magnetic energy spectrum. The higher-resolution spectra are plotted in dashed lines and agree quite well with the lower-resolution runs. There is a disagreement in the small scales in the kinetic spectra, but this has a low impact on the global behavior. Table I contains time-averaged values for the kinetic and magnetic energies, as well as for the Reynolds and Maxwell stresses, which are defined, respectively, as $\alpha_{\text{Rey}} = \langle u_x u_y \rangle$ and $\alpha_{\text{Max}} = -\langle B_x B_y \rangle$. The quantities are computed with both resolutions for $\text{Re} = 12\,500$. In the remainder of this Letter, we adopt the lower resolution. Note that the lack
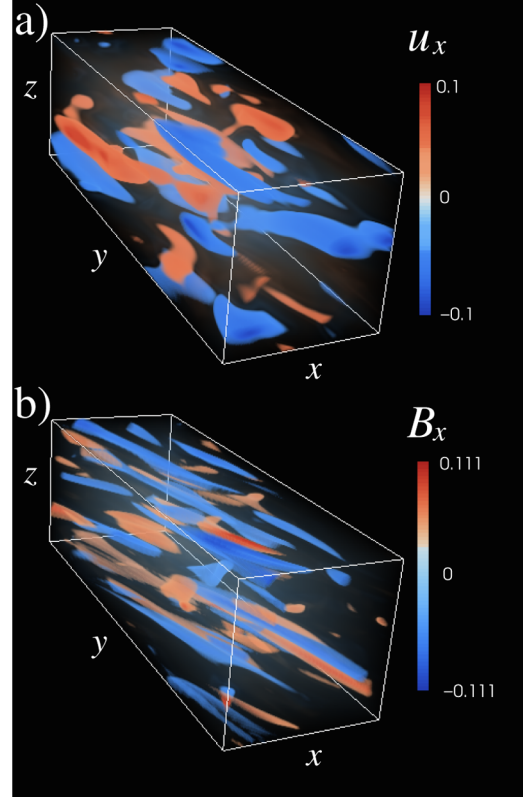

FIG. 1 (color online).   Volume rendering plots of u_x (a) and B_x (b) for a turbulent state at $\text{Re} = 3125$ and $R_m = 11\,000$.

of a properly defined inertial range in the power spectra indicates that the turbulence is not fully developed.

The turbulence shown in Fig. 1 eventually decays toward the laminar state. The decay time is a function of the initial perturbation and R_m. Figure 3 shows an example of a time series of the magnetic (red) and kinetic (black) energies for
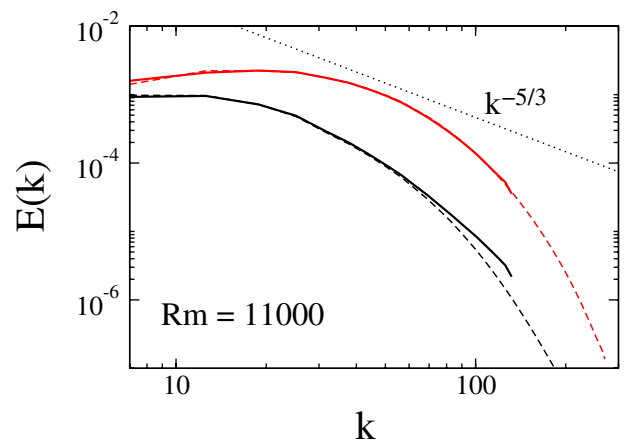

FIG. 2 (color online).   Time-averaged spectra of kinetic (solid black line) and magnetic (solid red line) energies for $R_e = 3125$ and $R_m = 11\,000$. The dashed lines represent the spectra computed with doubled resolution. The $k^{-5/3}$ dotted line is a guide to the eye.

TABLE I.   Time-averaged quantities for $R_m = 12\,500$.

Res.	$64 \times 128 \times 64$	$128 \times 256 \times 128$
$\langle E_{\text{kin}}\rangle_t$	2.11×10^{-3}	2.07×10^{-3}
$\langle E_{\text{mag}}\rangle_t$	8.45×10^{-3}	8.52×10^{-3}
$\langle \alpha_{\text{Rey}}\rangle_t$	5.22×10^{-4}	5.24×10^{-4}
$\langle \alpha_{\text{Max}}\rangle_t$	3.68×10^{-3}	3.69×10^{-3}

$Re = 3125$ and $R_m = 12\,500$. After more than 6000 time
units, the turbulence suddenly decays and the system con-
verges to the laminar state.

The high variability of the turbulence decay times as a
function of the initial condition for a given R_m suggests the
presence of a chaotic saddle in the phase space [2,20]. The
role of chaotic saddles in generating chaotic transients in
dynamical systems is well known, and, as mentioned in
previous studies on spatiotemporal systems [21], the aver-
age lifetime of the chaotic transients grows as a function of
the fractal dimension of the stable manifold of the chaotic
saddle. In order to obtain the average lifetime $\tau(R_m)$ of the
turbulent transients, we compute $P(t)$, the probability of
finding a turbulent state at time t. For each value of R_m,
$P(t)$ is obtained from a set of more than 50 and up to 90
different simulations. Each simulation has a turbulent state
as the initial condition, and the simulation stops when
either $t = 1000$ or the laminar state is reached, generating
a data set of relaminarization times. The turbulent initial
conditions are taken from a long run at $R_m = 13\,000$. To
ensure that the initial conditions represent uncorrelated
states, they were selected from the referred long turbulent
run with 500 shear-time units apart from each other.
Figure 4 shows the logarithm of $P(t)$ as a function of t
for $Re = 3125$ and a range of different values of R_m. The
straight lines are least square fits that represent the relation
$P(t, R_m) = \exp[-t/\tau(R_m)]$, which is expected for tran-
sients due to chaotic saddles [2,22]. The values of $1/\tau$

are readily obtained from the slopes of the fitted lines. As
R_m increases, the slopes decrease, reflecting longer decay
times. The convergence is not very good for the lowest
values of R_m, probably due to a lower fractal dimension of
the stable manifold of the chaotic saddle, which implies
that a much higher number of initial conditions would be
necessary to obtain better statistics.

A graph of τ versus R_m is plotted in Fig. 5. The top panel
shows the data in linear scales and reveals that the inverse
lifetime can be well fitted with an exponential law as a
function of R_m. The dashed curve is given by $1/\tau =
\exp[5.3 - (1.18 \times 10^{-3})R_m]$. The lower panel shows the
same data in log-linear scales. These results indicate that
the turbulence lifetime follows a type-II supertransient law
[17] as a function of R_m. In this case, there is no apparent
transition to sustained turbulence, and the laminar state is
the global attractor. If there were a critical R_m for transition
to sustained turbulence via a crisis mechanism, a linear
decay of $1/\tau$ with R_m would have been expected [3]. The
results for a linear fit using only the last four values of R_m
are given by the dotted blue line in Fig. 5(a), which
represents $1/\tau = 0.0144 - 1.273 \times 10^{-6}R_m$. This fit pro-
vides a critical point $R_{m_c} \sim 11\,312$, which is clearly in-
correct, since Fig. 3 shows that even at $R_m = 12\,500$ the
turbulence is still decaying.

Note that the high values of Re and R_m shown in this
Letter are based on the shear rate S. If the Reynolds
numbers are computed based on the root mean square
velocity, one obtains a maximum of $Re \sim 180$ and $R_m \sim
650$. Thus, it has been shown that, at least for intermediate
Reynolds numbers and magnetic Prandtl numbers greater
than 1, MHD turbulence in the incompressible shearing
box equations with zero net magnetic flux is not self-
sustained. The average decay time τ follows a type-II
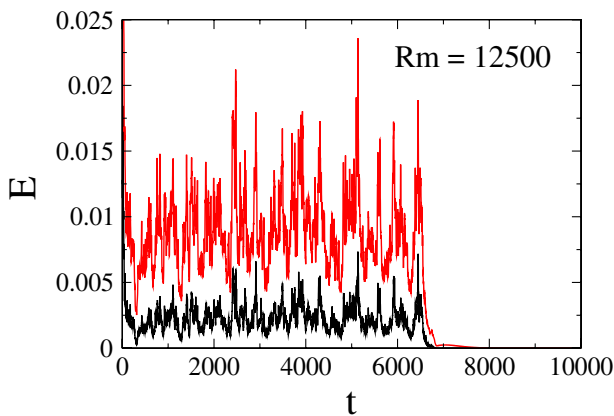supertransient law, with an exponential increase of τ as a

FIG. 3 (color online).   Time series of the magnetic (red line)
and kinetic (black line) energies showing transient turbulence for
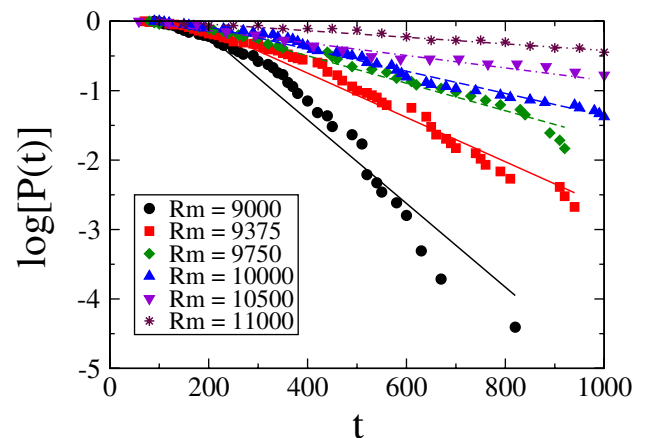$R_e = 3125$ and $R_m = 12\,500$.

FIG. 4 (color online).   The logarithm of $P(t)$, the probability of
turbulent lifetime $\geq t$, as a function of t for $R_e = 3125$ and six
different values of R_m; the straight lines are fitted with
$P(t, R_m) = \exp[-t/\tau(R_m)]$, where τ is the characteristic turbu-
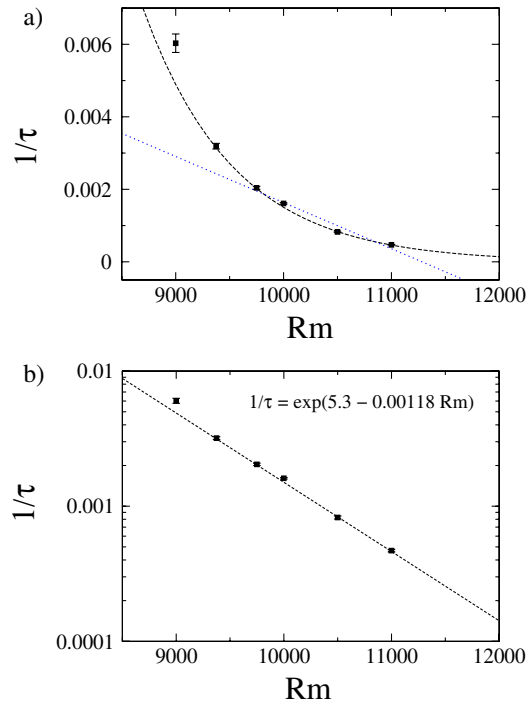lence lifetime.

FIG. 5 (color online).   The characteristic turbulent lifetimes as a function of R_m, obtained from fits to the data shown in Fig. 4; (a) linear scales, the dashed curve is an exponential fit, and the dotted straight line is a linear fit using only the last four points; (b) the same data as in (a) but in log-linear scales, with the exponential fit.

function of the magnetic Reynolds number. This is in accordance with results found for subcritical transitions to turbulence in hydrodynamic flows such as the pipe flow [2] and plane-Couette flow [1], where turbulence seems to be always transient. Supposing that our conclusions can be extended to realistic values of the Reynolds numbers, this suggests that the transient dynamics is much more relevant to the accretion problem than the final attractor.

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