Holographic Berezinskii-Kosterlitz-Thouless Transitions

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(Received 2 March 2010; published 20 July 2010)

We find the first example of a quantum Berenzinskii-Kosterlitz-Thouless (BKT) phase transition in two spatial dimensions via holography. This transition occurs in the $D3/D5$ system at nonzero density and magnetic field. At any nonzero temperature, the BKT scaling is destroyed and the transition becomes second order with mean-field exponents. We go on to conjecture about the generality of quantum BKT transitions in two spatial dimensions.

DOI: 10.1103/PhysRevLett.105.041601 PACS numbers: 11.25.Tq, 11.25.Uv, 64.70.Tg

Introduction.—Holography [1–3] has become an important tool in the investigation of strongly coupled systems. Despite being restricted to a special class of theories—those with simple gravitational duals—the technique has found interesting applications to the physics of the quark-gluon plasma [4]. Recently, there have been attempts to use holographic models to approach condensed matter systems—like the Fermi gas at unitarity [5,6], superfluids [7], and non-Fermi liquids [8–11]—with various degrees of success.

Phase transitions are a central physical concept that can be studied holographically. They are present in many holographic models and frequently allow simple geometric interpretations. However, most holographic phase transitions are either first-order [12] or second-order with mean-field exponents. The mean-field behavior arises from the large N parameter which suppresses quantum fluctuations in the gravity theory and allows the latter to be treated semiclassically. On the other hand, most interesting questions are usually beyond the mean-field approximation [13], or beyond the Landau-Ginzburg-Wilson paradigm altogether [14]. It is important to investigate if the holographic method can be used to study these non-mean-field phase transitions [15,16].

In this Letter, we show that, even within the confines of large-N field theories, another type of phase transition is possible—namely, those with the scaling behavior of the Berezinskii-Kosterlitz-Thouless (BKT) phase transition [17,18]. Recall that in the BKT phase transition, the order parameter scales as $\exp(-c/\sqrt{T_c - T})$ near the critical temperature T_c [19]. In our case, interestingly, the phase transition is a quantum phase transition, occurring at zero temperature in $2+1$ spacetime dimensions. The explicit example we consider is a $2+1$ dimensional theory at a finite density d of a conserved charge and a magnetic field B. At a particular value of the "filling fraction" $\nu = d/B$, the system suffers a transition to a broken symmetry state, and the scaling of the order parameter is the same as in a BKT phase transition, $\sigma \sim \exp(-c/\sqrt{\nu_c - \nu})$. We shall call this phase transition the "holographic BKT phase transition," although it happens in a context different from the original BKT phase transition. The BKT scaling occurs in quantum mechanics with a $1/r^2$ potential, and has been speculated to describe the chiral phase transition in QCD with large number of flavors [20].

On the gravitational side, the transition occurs due to the violation of the Breitenlohner-Freedman (BF) bound [21] in the infrared region by the scalar field dual to the order parameter. As shown in [20], this violation was expected to produce BKT scaling. This work represents the first time that the BF bound has been violated in a controlled setting. Although this mechanism seems similar to that of a second-order Landau phase transition, the field in the bulk actually represents an infinite tower of states in the boundary quantum field theory. At the phase transition, an infinite number of field theory modes become unstable at the same time—an extremely unnatural situation within the Landau theory.

The gravitational description of the holographic BKT transition is given in terms of a probe brane minimizing its worldvolume action in a fixed geometry. In Ref. [22], three of us studied one such probe describing a $3+1$ dimensional field theory. In $3+1$ dimensions the competition between finite density and the magnetic field gave rise to a second-order transition with mean-field scaling. (See Refs. [22,23] for technical details and a discussion of the related literature.) In this work, we will show that the same setup in $2+1$ dimensions gives rise to BKT scaling. The fact that the density and the magnetic field have the same mass dimension in $2+1$ dimensions will be crucial.

The $D3/D5$ system.—We write the $AdS_5 \times S^5$ background metric as

$$g = (r^2 + y^2)(-dt^2 + d\vec{x}^2) + \frac{1}{r^2 + y^2}(dr^2 + r^2 d\Omega_2^2 + dy^2 + y^2 d\Omega_2^2), \tag{1}$$

where $\vec{x}$ is a spatial three-vector and $d\Omega_2^2$ is the metric of a unit-radius two-sphere. In these coordinates the boundary of the AdS_5 is located at $r^2 + y^2 \to \infty$ and the horizon at $y = r = 0$.

We consider N_f probe $D5$ branes in this geometry, making an ansatz that the probes wrap the first two-sphere, the "radial coordinate" r, time, and two spatial directions. The branes have a profile parameterized by $y = y(r)$. The dual field theory is $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory at large 't Hooft coupling λ coupled to N_f fundamental hypermultiplets along a $2+1$-dimensional defect. This theory has a $U(1) \times SU(2)_1 \times SU(2)_2$ global symmetry, where the two $SU(2)$ factors are chiral R symmetries. The field y is dual to a condensate of the field theory which can spontaneously break the second $SU(2)$ factor. We are interested in the chiral phase transition between $y = 0$ and $y \neq 0$ configurations.

We turn on a density of the $U(1)$ flavor charge and a magnetic field coupled to this charge. On the gravity side, we turn on a field strength for the $U(1)$ gauge field dual to the current, $F = A_t'(r)dt \wedge dr + Bdx^1 \wedge dx^2$. The radial electric field A_t' is supported by charge behind the AdS horizon, so these branes extend to this horizon.

The Dirac-Born-Infeld (DBI) action density for these probe branes takes the form

$$S = -\mathcal{N} \int dr r^2 \sqrt{1 + y'^2 - A_t'^2} \sqrt{1 + \frac{B^2}{(r^2 + y^2)^2}}. \quad (2)$$

where $\mathcal{N} = \sqrt{\lambda} N_f N_c/(2\pi^3)$. Since A_t only appears through derivatives, $d \equiv \delta S/\delta A_t'$—the charge density—is r independent. The action at fixed density is obtained by Legendre transforming Eq. (2) with respect to A_t',

$$\tilde{S} = -\int dr \sqrt{1 + y'^2} \sqrt{d^2 + \mathcal{N}^2 r^4 + \frac{\mathcal{N}^2 r^4 B^2}{(r^2 + y^2)^2}}. \quad (3)$$

The onset of the phase transition can be found by analyzing the stability of small perturbations around $y = 0$, which are described by the quadratic part of (3),

$$L \sim -\frac{\mathcal{N}}{2} \sqrt{\rho^2 + B^2 + r^4} y'^2 + \frac{\mathcal{N} B^2 y^2}{r^2 \sqrt{\rho^2 + B^2 + r^4}}, \quad (4)$$

where $\rho = d/\mathcal{N}$. We pause to note two features of this Lagrangian. Near the boundary at large r, y/r fluctuates as a scalar with mass squared $m^2 = -2$ in AdS_4. However, at small r (the IR of the dual theory), y/r behaves like a scalar in AdS_2 with $m^2 = -2B^2/(B^2 + \rho^2)$. If $\rho/B < \sqrt{7}$, then the mass of y/r in the IR region is below the Breitenlohner-Freedman bound of stability for the effective AdS_2, $m_{\mathrm{BF}}^2 = -1/4$. In this case the trivial embedding $y = 0$ is unstable and the ground state should instead have $y \neq 0$. Thus, the chiral phase transition occurs at the filling fraction

$$\nu_c = \frac{d}{B_c} = \mathcal{N} \frac{\rho}{B_c} = \frac{\sqrt{7} \lambda N_f N_c}{2\pi^3}. \quad (5)$$

In the broken phase, the condensate can be found by solving the equation for the embedding with appropriate boundary conditions. We are interested in the critical behavior near the phase transition. In this regime, we find the embedding by matching the solutions in two regions: a small-r nonlinear core and a large-r linear tail.

In the core region, we can set $B^2 = \rho^2/7$, and neglect r^4 compared to ρ^2. The action then becomes

$$S \sim -\mathcal{N} \rho \int dr \sqrt{1 + y'^2} \sqrt{1 + \frac{r^4}{7(r^2 + y^2)^2}}. \quad (6)$$

Our brane embeddings, which extremize this action and obey the boundary condition $y(0) = 0$, form a one-parameter family of solutions related by scaling: $y_\xi(r) = \xi f(r/\xi)$, where $y = f(r)$ is one particular embedding. At large r, any solution f has the asymptotic form

$$f(r) = \sqrt{r}(-a_0 + a_1 \log r), \qquad r \gg 1. \quad (7)$$

Numerically solving the equation of motion by shooting, we find a solution f with $a_0 = -.211$ and $a_1 = .0585$. A general embedding in the core then has the asymptotics

$$y_\xi(r) = a_1 \sqrt{\xi r} \log\left(\frac{r}{r_0}\right), \qquad r \gg r_0 = \xi e^{a_0/a_1}. \quad (8)$$

It would be incorrect to use this expression too far away from the core: the terms neglected in the core action Eq. (6) become important. But in the linear regime y/r, $y' \ll 1$, the two independent solutions for y can be found without additional approximation,

$$f_\pm(u) = u_2^{(1 \pm i\alpha)/2} {}_2F_1\left(\frac{1 \pm i\alpha}{8}, \frac{3 \pm i\alpha}{8}, 1 \pm \frac{i\alpha}{4}, -u^4\right), \quad (9)$$

$$\alpha = \frac{\sqrt{\rho_c^2 - \rho^2}}{\sqrt{\rho^2 + B^2}}, \qquad u = \frac{r}{(\rho^2 + B^2)^{1/4}}. \quad (10)$$

When the filling fraction is just below the phase transition, α is real and small. From the two solutions in Eq. (9), we construct linear combinations that asymptote to $1/u$ near the boundary,

$$f_n(u) = c_+ f_+(u) + c_- f_-(u) \to \frac{1}{u}, \qquad u \to \infty. \quad (11)$$

This choice amounts to choosing zero bare mass for the dual flavor. The coefficient of the $1/u$ term is proportional to the condensate of the dual theory. Denoting the condensate as σ, the solution in the linear regime is $y(r) = -(\rho^2 + B^2)^{-1/4} \sigma f_n(u)$.

For small u and small α, f_n can be expanded

$$f_n(u) \sim \left[\frac{C_1}{i\alpha} + C_2\right] u^{(1-i\alpha)/2} + \left[-\frac{C_1}{i\alpha} + C_2\right] u^{(1-i\alpha)/2}$$
$$= \sqrt{u}\left[2C_2 \cos\left(\frac{\alpha}{2} \ln u\right) - \frac{2C_1}{\alpha} \sin\left(\frac{\alpha}{2} \ln u\right)\right], \quad (12)$$

where

$$C_1 = \frac{\Gamma^2(1/4)}{2^{3/4}\pi^{3/2}}, \qquad C_2 = \frac{2^{1/4}\Gamma^2(5/4)(\ln 256 - \pi)}{\pi^{3/2}}. \tag{13}$$

To leading order in α, we rewrite Eq. (12) as

$$y(r) = \frac{2C_1}{(\rho^2 + B^2)^{3/8}} \sigma \frac{\sqrt{r}}{\alpha} \sin\left(\frac{\alpha}{2} \ln\frac{r}{r_1}\right), \tag{14}$$

$$r_1 = (\rho^2 + B^2)^{1/4} \exp\left(\frac{2C_2}{C_1}\right). \tag{15}$$

To match the core and linear solutions at $r = \xi$, the argument of the sin in Eq. (14) has to make π between $r_0 \sim \xi$ and $r_1 \sim \sqrt{B}$. To exponential accuracy, we then have $\xi \sim e^{-2\pi/\alpha}$. Comparing Eqs. (8) and (14), we find

$$\sigma \sim \sqrt{\xi} \sim e^{-\pi/\alpha}. \tag{16}$$

Thus, the phase transition in the $D3/D5$ system obeys BKT scaling. The exponent as well as the prefactors can also be found by matching both y and y' at ξ,

$$\sigma = -\mathcal{N}\frac{a_1}{C_1}\sqrt{\rho^2 + B^2} \exp\left[-\frac{\pi}{\alpha} + \frac{C_2}{C_1} - \frac{a_0}{2a_1} + O(\alpha)\right]. \tag{17}$$

We also compare this result to numerical data in Fig. 1.

The way we matched the core and the tail of the embedding makes it clear that there exists an infinite set of "Efimov extrema," which we name for their resemblance to Efimov states [24]. Indeed, the most general match has an argument $n\pi$ in the sin of Eq. (14) between $r \sim r_0$ and $r \sim r_1$, for n a positive integer. Matching y and y' at $r = \xi$, we find a condensate corresponding to these extrema of $\sigma_n \sim e^{-n\pi/\alpha}$, where σ_1 is the condensate for the simplest embedding above. The additional extrema are not ground states, however: denoting the free energy of the trivial embedding as F_0, the free energy of the n-th extremum goes like $F_0 - F_n \sim e^{-2\pi n/\alpha}$, so the $n = 1$ embedding is the ground state. The infinite tower of extrema, spaced by the same factor $\exp(-2\pi/\alpha)$, is reminiscent of the Efimov states.

We next examine the $D3/D5$ system at finite temperature T. To do so, we embed the $D5$ probes in an AdS-Schwarzschild black hole geometry. The embedding equations can now only be solved numerically. However, at any temperature the magnetic field does not contribute in the effective AdS_2 region in the IR. We could therefore predict that the BKT scaling is lost for $T > 0$. We numerically solved the embedding equations at a number of different temperatures. (The techniques employed are similar to those used in Ref. [22].) We plot some of our results for the condensate in Fig. 1. At fixed finite temperature, we find that there is still a chiral symmetry breaking transition but that it is a mean-field second-order one. Far away from the transition, we recover the exponential BKT scaling at zero temperature.

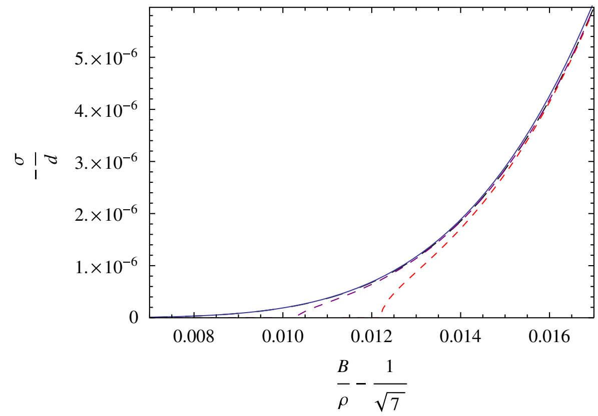

FIG. 1 (color online). A plot of the condensate as a function of magnetic field at zero and finite temperature near the zero-temperature transition. The dashed black line indicates zero-temperature numerical data and the solid blue line our prediction, Eq. (17), computed further to $\mathcal{O}(\alpha^3)$. The color dashed lines represent numerical data at temperatures $T = \frac{2}{\pi}\rho^{1/2} \times 10^{-11}$ (left) and $T = \frac{2}{\pi}\rho^{1/2} \times 10^{-10}$ (right). At any nonzero temperature, the condensate scales with a mean-field exponent near the transition, asymptoting to the BKT scaling further away.

Lifshitz systems.—In the example that has been just considered, the BKT scaling arises from an IR AdS_2 region and the existence of a scalar with mass squared crossing the BF bound at the transition. We may expect the BKT scaling to be present in many other cases. We now demonstrate that the BKT scaling indeed controls the critical behavior in phase transitions in a rather large class of models.

Our models are built on the so-called Lifshitz geometries [25], which are invariant under anisotropic scaling, $t \to \lambda^z t$, $\vec{x} \to \lambda \vec{x}$. Under holography, a geometry of this type is conjectured to be dual to a scale-invariant boundary field theory with dynamic critical exponent z. Constructing explicit Lifshitz solutions from string theory is quite involved [11]; our approach here is phenomenological. Nevertheless, once a few conditions (to be specified below) are satisfied, the BKT scaling is rather generic—which seems to indicate that it should occur in string-theoretical realizations of Lifshitz geometries as well.

We consider probe branes with an induced metric

$$P[g] = -(r^2 + y^2)^z dt^2 + (r^2 + y^2) d\vec{x}^2 + \frac{dr^2(1 + y'^2) + r^2 g_{\rm int}}{r^2 + y^2} \tag{18}$$

and constant dilaton. Here, r is the "radial coordinate" of the geometry, $\vec{x} \in \mathbb{R}^D$ is a spatial vector, $g_{\rm int}$ is the metric on a wrapped space of dimension k, and $y = y(r)$ is the profile of the branes. The $D3/D5$ system corresponds to $z = 1$ and $D = k = 2$.

We now turn on a density and magnetic field in the field theory, dual to $U(1)$ field strengths on the branes. Assuming that Chern-Simons terms play no role, the action of the probes at fixed density is

$$\tilde{S} = -\mathcal{N}_z \int drR^{z-1}\sqrt{1+y'^2}\sqrt{\rho^2 + \frac{r^{2k}}{R^{2(k-D)}}\left(1+\frac{B^2}{R^4}\right)}, \tag{19}$$

where we have defined $R^2 \equiv r^2 + y^2$, $\mathcal{N}_z$ is a prefactor, and ρ is a rescaled density $d = \mathcal{N}_z \rho$.

As before, we attempt to find the location of the phase transition by looking at small fluctuations around the trivial embedding $y(r) = 0$. In the UV, y/r is a massive scalar in AdS_{z+D+1}, corresponding to an operator with dimension $\Delta = (z+D)/2 + \sqrt{(z+2)^2 - 4k}/2 + 1$ in field theory. At small r, y/r fluctuates in two different ways depending on the value of D. For two spatial dimensions, y/r fluctuates as a $m^2 = -kB^2/(B^2+\rho^2)$ scalar in AdS_{z+1}; for $D > 2$, y/r behaves like a $m^2 = 0$ scalar in AdS_{z+1}. The holographic BKT transition is then only possible for $D = 2$, where a density and magnetic field have the same dimension. The transition occurs at

$$\frac{\rho}{B_c} = \frac{\sqrt{4k - z^2}}{z}, \tag{20}$$

for which the bound $4k > z^2$ must also be satisfied. This is also the bound for which a magnetic field at zero density will break chiral symmetry.

Using the same method as employed in the $D3/D5$ case, we study zero bare mass by taking y to be normalizable at large r with $y \sim 1/r^{\Delta-1}$. Holographic renormalization relates the coefficient of this term to the condensate σ, which we find to be

$$\sigma = -\mathcal{N}_z \frac{a_1}{C_1}(\rho^2 + B^2)^{\Delta/4}(\Delta - 1) \times \exp\left[-\frac{\pi}{\alpha} + \frac{zC_2}{C_1} - \frac{za_0}{2a_1}\right], \tag{21}$$

where C_1, C_2 are complicated constants that depend on Δ and z, and a_0, a_1 are the asymptotic data of a core solution to which we match. We can therefore find BKT scaling for any dynamical exponent, provided that the magnetic field can break chiral symmetry.

Discussion.—We therefore see that holographic phase transitions in $2+1$ dimensions, with the ratio of the magnetic field and the density as a control parameter, may generally be of the BKT type. It would be extremely interesting to find such a BKT transition away from the large N and strong coupling limits.

The existence of an infinite number of "Efimov vacua" is clearly related to the breakdown of the Landau's theory. In Landau's theory of phase transitions, the order parameter is the only low-energy degree of freedom. In our case, the role of the order parameter is played by the whole function $y(r)$, where r is usually interpreted as the energy scale. One needs to further investigate the low-energy effective theory of the $D3/D5$ system.

The authors thank Carlos Hoyos and Steve Paik for many useful discussions. This work was supported, in part, by the U.S. Department of Energy under Grant Numbers DE-FG02-96ER40956 and DE-FG02-00ER41132 and by an RRF grant from University of Washington.

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