Interaction-Induced Criticality in $\mathbb{Z}_2$ Topological Insulators

P. M. Ostrovsky,[1,2] I. V. Gornyi,[1,3] and A. D. Mirlin[1,4,5]

[1] *Institut für Nanotechnologie, Karlsruhe Institute of Technology, 76021 Karlsruhe, Germany*
[2] *L. D. Landau Institute for Theoretical Physics RAS, 119334 Moscow, Russia*
[3] *A. F. Ioffe Physico-Technical Institute, 194021 St. Petersburg, Russia*
[4] *Institut für Theorie der kondensierten Materie, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany*
[5] *Petersburg Nuclear Physics Institute, 188300 St. Petersburg, Russia*

(Received 1 December 2009; published 16 July 2010)

We study interaction effects in topological insulators with strong spin-orbit coupling. We find that the interplay of nontrivial topology and Coulomb repulsion induces a novel critical state on the surface of a three-dimensional topological insulator. Remarkably, this interaction-induced criticality, characterized by a universal value of conductivity, emerges without any adjustable parameters. Further, we predict a direct quantum-spin-Hall transition in two dimensions that occurs via a similar critical state.

DOI: 10.1103/PhysRevLett.105.036803 PACS numbers: 73.20.−r, 73.43.Nq

Critical phenomena and quantum phase transitions are paradigmatic concepts in modern condensed matter physics. The universality of critical phenomena has been studied both in the area of strongly correlated systems and in mesoscopics. A central example in the field of mesoscopic physics is the localization-delocalization (metal-insulator) quantum phase transition driven by disorder—the Anderson transition [1]. Although the notion of localization appeared half a century ago, this field is still full of surprising new developments. The most recent arenas where novel peculiar localization phenomena have been studied are graphene [2] and topological insulators [3–7], i.e., bulk insulators with delocalized (topologically protected) states on their surface.

It is now established that disordered electronic systems can be classified into 10 symmetry classes [8] (for review see Ref. [1]). The localization properties are determined by the symmetry class and dimensionality of the system. The critical behavior depends also on the underlying topology. This is particularly relevant for topological insulators [3–7,9–11]. The famous example of a topological insulator (TI) is a two-dimensional (2D) system on one of the quantum Hall plateaus in the integer quantum Hall effect (QHE) [12]. Such a system is characterized by an integer (Chern number) which counts the edge states. The integer QHE edge is thus a topologically protected one-dimensional (1D) conductor realizing the group $\mathbb{Z}$.

Another ($\mathbb{Z}_2$) class of TIs [3–5] can be realized in systems with strong spin-orbit (SO) interaction and without magnetic field—and was discovered in HgTe/HgCdTe structures in Ref. [6] (see also Ref. [9]). Such systems were found to possess two distinct insulating phases, both having a gap in the bulk electron spectrum but differing by edge properties. While the normal insulating phase has no edge states, the topologically nontrivial insulator is characterized by a pair of mutually time-reversed delocalized edge modes penetrating the bulk gap. Such a state shows the quantum-spin-Hall (QSH) effect which was theoretically predicted in a model system of graphene with SO coupling [3,13,14]. The transition between the two topologically nonequivalent phases (ordinary and QSH insulators) is driven by inverting the band gap [4].

A related three-dimensional (3D) $\mathbb{Z}_2$ TI was discovered in Ref. [7] where crystals of $Bi_{1-x}Sb_x$ were investigated. The boundary in this case gives rise to a 2D topologically protected metal. Similarly to 2D TIs, the inversion of the 3D band gap induces an odd number of the surface 2D modes [15]. These states in BiSb have been studied experimentally in Refs. [7,10]. Other examples of 3D TIs include BiTe and BiSe systems [11]. We overview the classification [16] of TIs in the supporting material [17].

In this Letter, we consider the effect of interactions on $\mathbb{Z}_2$ TIs of the symplectic symmetry class, characteristic to systems with strong SO interaction. We predict a novel critical state which emerges due to the interplay of nontrivial topology and the Coulomb interaction.

Let us start with reviewing the localization properties of 2D systems of symplectic symmetry class AII without Coulomb interaction. In conventional SO systems (e.g., semiconductors with SO scattering), there are two phases: metal and insulator with the Anderson transition between them [Fig. 1(a)]. A qualitatively different situation occurs in a single species of massless Dirac fermions in a random scalar potential. This system also belongs to the symplectic symmetry class but its metallic phase is "topologically protected" whatever disorder strength. In terms of scaling, this means a positive beta function, $\beta(g) = dg/d\ln L > 0$, for small dimensionless (in units e^2/h) conductivity g [Fig. 1(b)]. This topologically protected state has been recently predicted for disordered graphene with no spin and no valley mixing [18,19] (see also Ref. [17] for an alternative proof). The absence of localization in this model has been confirmed in numerical simulations [20]. The scaling function has been found in Ref. [20] to be strictly positive, implying a flow towards the "supermetal" fixed point [21] [see Fig. 1(b)]. While a genuine single

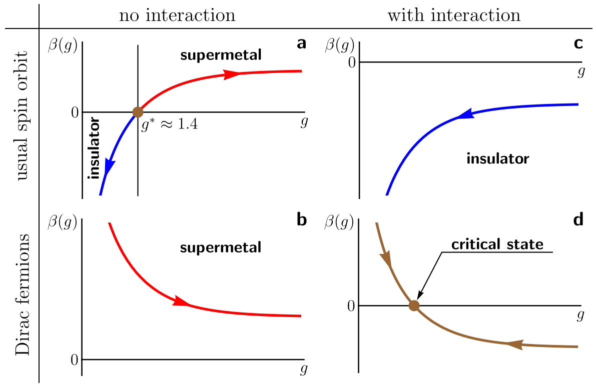

FIG. 1 (color online). Schematic scaling functions for the conductivity of 2D disordered systems of symplectic symmetry class. The plotted beta functions $\beta(g) = dg/d\ln L$ determine the flow of the dimensionless conductivity g with increasing system size L (as indicated by the arrows). The upper two panels show the beta functions for ordinary SO systems which are not topologically protected; the lower two panels demonstrate the scaling for topologically protected Dirac fermions (left: no interaction; right: Coulomb interaction included). In the interacting case the number of independent flavors is $N = 1$.

Dirac fermion cannot be realized in a truly 2D microscopic theory because of the "fermion doubling" problem, it emerges on the surface of a 3D TI [15].

The 3D TIs are characterized by the inverted sign of the gap (band inversion). This generates the surface states, as was pointed out in Ref. [22]. The effective 2D surface Hamiltonian has a Rashba form (see the derivation in Ref. [17]) and describes a single species of 2D massless Dirac particles (cf. Ref. [22(c)]). It is thus analogous to the Hamiltonian of graphene with just a single valley. In the absence of interaction, the conductivity of the disordered surface of a 3D TI therefore scales to infinity with increasing the system size [Fig. 1(b)].

Let us now "turn on" the Coulomb interaction between electrons. Since a TI is characterized by the presence of propagating surface modes, its robustness with respect to interactions means that interactions do not localize the boundary states. At this point it is worth recalling the celebrated example of a 2D TI, the QHE insulator, in which the Coulomb interaction cannot destroy the chiral 1D modes on the edge of a 2D sample. Furthermore, two consequent QHE TIs (plateaus) are separated by a delocalized (critical) 2D state. Since the TI phases are robust, the interaction is not capable of localizing electrons in this 2D state [23]: a delocalized bulk state is necessary for changing the number of the edge modes at the QHE transition. The observation of the QHE thus provides an experimental proof of the robustness of TIs.

Assuming that the interaction (characterized by the dimensionless parameter $r_s \sim e^2/\hbar v_F$) is not too strong, delocalized states in $\mathbb{Z}_2$ TIs are not destroyed by the interaction either. Arguments in favor of the stability of $\mathbb{Z}_2$ TIs with respect to interactions were given in Refs. [3,5,24]. Below we demonstrate the $\mathbb{Z}_2$ topological order and discuss its implications in 2D and 3D interacting systems in the presence of disorder.

We first consider the interacting massless Dirac electrons on the surface of a 3D TI. Without interaction, the surface states are delocalized in the presence of arbitrarily strong potential disorder. In the supporting material [17] we demonstrate that a not too strong interaction cannot fully localize the surface states. This is achieved by considering the TI of a hollow cylinder geometry threaded by half of the magnetic flux quantum. Our 2D problem then reduces to the quasi-1D model with an odd number of channels. Full 2D localization would contradict the known results on the absence of localization in such quasi-1D symplectic wires [25]. Since delocalization in quasi-1D geometry survives in the presence of interaction, this is also true for the 2D interacting Dirac electrons on the surface of a 3D TI. This consideration is valid when the interaction does not lead to a spontaneous symmetry breaking, which is true for not too strong interaction $r_s \lesssim 1$ [17].

Can the topologically protected 2D state be a supermetal ($g \to \infty$) as in the noninteracting case? To answer this question we employ the perturbative renormalization group applicable for large conductivity $g \gg 1$. It is well known that in a 2D diffusive system the interaction leads to logarithmic corrections to the conductivity [26]. These corrections (together with the interference-induced ones) can be summed up with the use of renormalization group technique [27–31]. The one-loop equation for renormalization of the conductivity in the symplectic class with Coulomb interaction has the following form:

$$\beta(g) = \frac{dg}{d\ln L} = \frac{N}{2} - 1 + (N^2 - 1)\mathcal{F}, \tag{1}$$

where N is the number of degenerate species ("flavors": spin, isospin, ...) and L is the system size [17].

The first term, $N/2$, describes the effect of weak antilocalization due to disorder (this term exists also in the absence of interaction) for N parallel conductors. The second term, -1, is induced by the Coulomb interaction in the singlet channel and has a localizing effect: it suppresses the conductivity. The singlet interaction term does not depend on N since all flavors are involved in the screening of the Coulomb interaction [26]. The effective strength of the singlet interaction is therefore suppressed by the factor $1/N$ which compensates the number of parallel channels in the conductivity correction. The last term on the right-hand side of Eq. (1) is due to the interaction in the multiplet (in the flavor space) channel. This term yields a positive (antilocalizing) correction to the conductivity. The multiplet interaction parameter $\mathcal{F}$ is itself subject to renormalization [27].

In the degenerate case $N > 1$ (as, for example, in graphene with $N = 4$), the beta function (1) is positive corresponding to the supermetal phase. The situation is essentially different for 2D states on the surface of a 3D $\mathbb{Z}_2$ TI where we have a symplectic system with $N = 1$. According to Eq. (1), the negative interaction-induced term in $\beta(g)$ now dominates while the multiplet term is absent. Therefore, for $g \gg 1$ the conductance decreases upon renormalization. This means that due to interaction the supermetal fixed point becomes repulsive.

Thus, on one hand, at $g \gg 1$ we encounter the tendency to localization due to the interaction. This follows from Eq. (1) which yields $\beta(g) < 0$, i.e., (i) scaling towards smaller g on the side of large g. On the other hand, the states on the surface of the TI are topologically protected from the localization (for the interacting system, see the proof in Ref. [17]). This topological protection yields $\beta(g) > 0$ at small g, i.e., (ii) scaling towards higher g on the side of small g. The combination of (i) and (ii) leads unavoidably to the conclusion that at some point ($g \sim 1$) the beta function should cross zero [see Fig. 1(d)]. As a result, a critical point emerges due to the combined effect of interaction and topology [32]. We emphasize that this statement does not require the knowledge of the precise form of the beta function in the critical region. In other words, if the system can flow neither towards a supermetal ($g \to \infty$) nor to an insulator ($g \to 0$), it must flow to an intermediate fixed point ($g \sim 1$).

It is illuminating to draw an analogy with the two-channel symmetric Kondo model, where both weak- and strong-coupling fixed points are repulsive [33], implying a stable fixed point at intermediate coupling. Critical phases of interacting systems governed by intermediate-coupling fixed points have been also found, e.g., in other impurity models [34] and in spin liquids [35].

The topological protection reverses the sign of the β function at $g \sim 1$, similarly to the ordinary QHE [23]. This is encoded in the topological $\theta = \pi$ term in the effective low-energy theory—the interacting symplectic sigma model. However, our type of criticality differs from the QHE criticality which exists already without interactions. In our case, in the absence of the critical state in a noninteracting model, the criticality is inevitably established in the realistic interacting systems. This novel interaction-induced criticality is the major result of our Letter. Remarkably, the critical state emerges on the surface of a 3D TI without any adjustable parameters. This phenomenon can be thus called "self-organized quantum criticality."

Let us now return to 2D $\mathbb{Z}_2$ TIs. Without interaction, disorder was found to induce a metallic phase separating the two (QSH and ordinary) insulators [36]. The transition [1] between metal and any of the two insulators occurs at the critical value of conductivity $g = g^* \approx 1.4$; both transitions are believed to belong to the same universality class. For $g < g^*$ all bulk states are eventually localized in the limit of large system, while for $g > g^*$ the weak antilocalization specific to SO systems leads to the "supermetallic" state, $g \to \infty$. The schematic phase diagram for the noninteracting case is shown in Fig. 2(a).

What would change in this phase diagram when interaction is taken into account? The answer follows from Eq. (1). The 2D disordered QSH system contains only a single flavor, $N = 1$. Indeed, the SO coupling breaks the spin-rotational symmetry, whereas valleys are mixed by disorder. As a result, the supermetal does not survive in the presence of Coulomb interaction: at $g \gg 1$ the interaction-induced localization wins. This is analogous to the case of the surface of a 3D TI discussed above.

The edge of a 2D TI is protected from the full localization, as was discussed already in the pioneering works by Kane and Mele [3]. In the presence of interaction, the counterpropagating edge modes constitute the Luttinger liquid with disorder-induced backscattering forbidden by the time-reversal symmetry. This means that the topological distinction between the two insulating phases (ordinary and QSH insulator) is not destroyed by a not too strong interaction (see [17]), whereas the supermetallic phase separating them disappears. Therefore we conclude that the transition between two insulators occurs through an interaction-induced critical state [see Fig. 2(b)].

Existence of the proposed critical states can be verified in transport experiments on semiconductor structures with possible gap inversion. Specifically, measurements of surface transport in a 3D TI should reveal universal conductivity $\sim e^2/h$ as well as critical power-law dependence of two-point conductance on the distance between contacts. For 2D HgTe-based QSH structures we predict a quantum phase transition (bearing analogy with the QHE transition) with changing width of the quantum well, which can be observed in conductance measurements. The transition should possess critical properties described above, and the critical region should shrink with decreasing temperature according to a power law.

In the above, we have considered the experimentally relevant case of long-range Coulomb repulsion. The topological protection is not sensitive to the nature of the interaction, whereas the flow at high g is. For short-range interaction, the unity in the singlet term of Eq. (1) is replaced by the interaction constant $\gamma_s < 1$, which itself decreases upon renormalization. Therefore, for sufficiently weak short-range interaction the supermetallic fixed point remains stable and the critical state does not develop [17]. For the strong short-range interaction, the situation is more intricate, as, in contrast to the Coulomb case, the analysis requires the details of scaling flows at intermediate $g \sim 1$.

In conclusion, we have found that in the two types of systems with strong SO coupling the Coulomb interaction

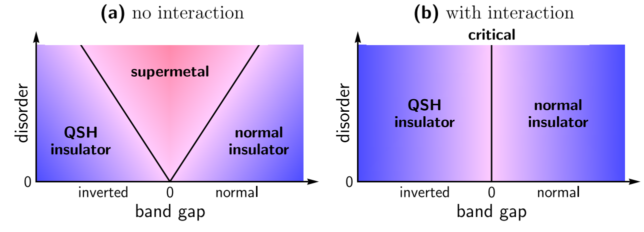

FIG. 2 (color online). Phase diagrams of a disordered 2D system demonstrating the QSH effect. (a) Noninteracting case. (b) Coulomb interaction included. Interaction "kills" the supermetal: the two insulators are separated by the critical line.

induces novel 2D critical states. This happens, first, at the boundary of 3D TIs and, second, in the bulk of 2D QSH systems, where the critical state separates the two topologically distinct insulating phases. In the first case the system can be described by a 2D interacting symplectic sigma model with the $\mathbb{Z}_2$ topological term. The two critical states have much in common: (i) symplectic symmetry, (ii) $\mathbb{Z}_2$ topological protection, (iii) interaction-induced criticality, and (iv) conductivity of order unity (probably universal). This suggests that the corresponding fixed points might be equivalent. The difference between the 3D and 2D cases is that on the surface of 3D TIs the criticality is "self-organized"—i.e., it emerges without any adjustable parameters—whereas in 2D TIs it requires fine-tuning of the parameter that controls the QSH transition. We propose transport experiments that can verify our prediction of novel quantum critical states.

We thank A. Altland, D. Bagrets, I. Burmistrov, F. Evers, M. Feigel'man, A. Finkelstein, V. Kagalovsky, D. Khmelnitskii, V. Kravtsov, I. Lerner, A. Ludwig, L. Molenkamp, I. Protopopov, and A. Rosch for discussions. The work was supported by DFG—CFN, EUROHORCS/ESF, and Rosnauka Grant No. 02.740.11.5072.

[1] F. Evers and A. D. Mirlin, Rev. Mod. Phys. **80**, 1355 (2008).
[2] A. H. Castro Neto *et al.*, Rev. Mod. Phys. **81**, 109 (2009).
[3] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005); **95**, 226801 (2005).
[4] B. A. Bernevig *et al.*, Science **314**, 1757 (2006); B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
[5] X.-L. Qi, T. L. Hughes, and S. C. Zhang, Phys. Rev. B **78**, 195424 (2008); Z. Wang, X.-L. Qi, and S.-C. Zhang, arXiv:1004.4229.
[6] M. König *et al.*, Science **318**, 766 (2007).
[7] D. Hsieh *et al.*, Nature (London) **452**, 970 (2008).
[8] M. R. Zirnbauer, J. Math. Phys. (N.Y.) **37**, 4986 (1996); A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).
[9] M. König *et al.*, J. Phys. Soc. Jpn. **77**, 031 007 (2008); A. Roth *et al.*, Science **325**, 294 (2009).
[10] D. Hsieh *et al.*, Science **323**, 919 (2009); P. Roushan *et al.*, Nature (London) **460**, 1106 (2009).
[11] D. Hsieh *et al.*, Nature (London) **460**, 1101 (2009); Y. L. Chen *et al.*, Science **325**, 178 (2009); Y. Xia *et al.*, Nature Phys. **5**, 398 (2009); H. Zhang *et al.*, *ibid.* **5**, 438 (2009).
[12] *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer, New York, 1987).
[13] L. Sheng *et al.*, Phys. Rev. Lett. **95**, 136602 (2005).
[14] A similar topological phenomenon has been earlier predicted in the context of superfluid helium-3 films in G. E. Volovik, JETP **67**, 1804 (1988); G. E. Volovik and V. M. Yakovenko, J. Phys. Condens. Matter **1**, 5263 (1989).
[15] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007); L. Fu and C. L. Kane, Phys. Rev. B **76**, 045302 (2007).
[16] A. P. Schnyder *et al.*, Phys. Rev. B **78**, 195125 (2008); in *Advances in Theoretical Physics*, edited by V. Lebedev and M. Feigelman, AIP Conf. Proc. No. 1134 (AIP, New York, 2009), p. 10; A. Yu. Kitaev, in *Advances in Theoretical Physics*, edited by V. Lebedev and M. Feigelman, AIP Conf. Proc. No. 1134 (AIP, New York, 2009), p. 22; S. Ryu *et al.*, New J. Phys. **12**, 065010 (2010).
[17] See supplementary material at http://link.aps.org/supplemental/10.1103/PhysRevLett.105.036803 for details.
[18] P. M. Ostrovsky, I. V. Gornyi, and A. D. Mirlin, Phys. Rev. Lett. **98**, 256801 (2007); Eur. Phys. J. Special Topics **148**, 63 (2007).
[19] S. Ryu *et al.*, Phys. Rev. Lett. **99**, 116601 (2007).
[20] J. H. Bardarson *et al.*, Phys. Rev. Lett. **99**, 106801 (2007); K. Nomura, M. Koshino, and S. Ryu, *ibid.* **99**, 146806 (2007).
[21] In Ref. [18] it was conjectured that the β function of Fig. 1(b) (positive at large and small g) may cross zero twice, yielding an additional attractive fixed point at $g \sim 1$. This type of criticality was not observed in the numerical simulation of Ref. [20]; see "Note added" in Ref. [18].
[22] (a) M. I. D'yakonov and A. V. Khaetskii, JETP Lett. **33**, 110 (1981); (b) V. A. Volkov and T. N. Pinsker, Sov. Phys. Solid State **23**, 1022 (1981); (c) B. A. Volkov and O. A. Pankratov, JETP Lett. **42**, 178 (1985).
[23] A. M. M. Pruisken and I. S. Burmistrov, Ann. Phys. (N.Y.) **322**, 1265 (2007).
[24] S.-S. Lee and S. Ryu, Phys. Rev. Lett. **100**, 186807 (2008).
[25] M. R. Zirnbauer, Phys. Rev. Lett. **69**, 1584 (1992); A. D. Mirlin *et al.*, Ann. Phys. (N.Y.) **236**, 325 (1994); T. Ando and H. Suzuura, J. Phys. Soc. Jpn. **71**, 2753 (2002); Y. Takane, *ibid.* **73**, 1430 (2004).
[26] B. L. Altshuler and A. G. Aronov, in *Electron-Electron Interactions in Disordered Conductors*, edited by A. L. Efros and M. Pollak (Elsevier, New York, 1985), p. 1.
[27] A. M. Finkelstein, Sov. Sci. Rev., Sect. A **14**, 1 (1990).
[28] D. Belitz and T. R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).
[29] C. Castellani *et al.*, Phys. Rev. B **30**, 527 (1984).
[30] C. Castellani *et al.*, Solid State Commun. **52**, 261 (1984); M. Ma and E. Fradkin, Phys. Rev. Lett. **56**, 1416 (1986).
[31] A. Punnoose and A. M. Finkel'stein, Phys. Rev. Lett. **88**, 016802 (2001); Science **310**, 289 (2005).
[32] Our findings should be distinguished from another type of interaction-induced criticality in disordered systems predicted in Ref. [31] for $N \gg 1$ and no SO coupling. In that case the interaction (the multiplet term $\propto \mathcal{F}$) overcomes localization, leading to a metal-insulator transition. We have just the opposite scenario of the formation of a critical state: the interaction tries to localize the system, but the topological protection prevents full localization.
[33] P. W. Anderson, J. Phys. C **3**, 2436 (1970); P. Nozières and A. Blandin, J. Phys. (Paris) **41**, 193 (1980).
[34] M. Vojta, Rep. Prog. Phys. **66**, 2069 (2003).
[35] M. Hermele *et al.*, Phys. Rev. B **70**, 214437 (2004).
[36] M. Onoda, Y. Avishai, and N. Nagaosa, Phys. Rev. Lett. **98**, 076802 (2007); S. Murakami *et al.*, Phys. Rev. B **76**, 205304 (2007); H. Obuse *et al.*, *ibid.* **76**, 075301 (2007); S. Ryu *et al.*, New J. Phys. **12**, 065005 (2010).