Families of Particles with Different Masses in PT-Symmetric Quantum Field Theory

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An elementary field-theoretic mechanism is proposed that allows one Lagrangian to describe a family of particles having different masses but otherwise similar physical properties. The mechanism relies on the observation that the Dyson-Schwinger equations derived from a Lagrangian can have many different but equally valid solutions. Nonunique solutions to the Dyson-Schwinger equations arise when the functional integral for the Green's functions of the quantum field theory converges in different pairs of Stokes' wedges in complex-field space, and the solutions are physically viable if the pairs of Stokes' wedges are $\mathcal{PT}$ symmetric.

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The standard model of elementary particles has three generations of fermions (leptons and quarks) whose masses range over several orders of magnitude. It is not known why there are three generations of masses and whether there are only three. This Letter proposes a field-theoretic mechanism that might explain the occurrence of generations of particles having different masses but otherwise similar physical properties: There might be just one Lagrangian (or Hamiltonian) to account for the properties of all these particles, but the functional integral constructed from this Lagrangian may have many different physical realizations depending on the boundary conditions on the path of integration in complex-field space. While the Dyson-Schwinger equations constructed from the functional integral are unique, the solution to these equations is not unique. The number of distinct solutions to the Dyson-Schwinger equations equals the number of pairs of complex Stokes' wedges in function space in which the boundary conditions on the functional integration can be imposed. For each pair of Stokes' wedges there corresponds a different field theory.

Z. Guralnik *et al.* [1] first recognized that for functional integrals, inequivalent classes of contours associated with different complex boundary conditions give rise to nonunique solutions to the Dyson-Schwinger equations. They argued that multiple solutions might account for inequivalent θ vacua. The key point of the current Letter is that the pairs of Stokes' wedges in which the integration contours terminate must be oriented in a $\mathcal{PT}$-symmetric fashion in complex-field space. If this is the case, there is strong evidence that the corresponding field theory will be physically acceptable; that is, the masses (poles of the Green's functions) will be real and the theory will be unitary. The mechanism proposed here is field theoretic, but its application is not restricted to elementary particle physics. Experiments on $\mathcal{PT}$-symmetric optical waveguides [2,3] and on $\mathcal{PT}$-symmetric diffusion [4] have been reported recently.

The conjecture discussed in this Letter stems from recent research on $\mathcal{PT}$ quantum mechanics, where it has been shown that the $\mathcal{PT}$-symmetric Hamiltonians

$$H = p^2 + q^2(iq)^\epsilon \qquad (\epsilon \geq 0) \tag{1}$$

all have real positive spectra [5,6]. Each of these Hamiltonians defines a conventional quantum theory with a Hilbert space having a positive inner product [7]. The time-evolution operator $U = e^{-iHt}$ is unitary and thus probability is conserved. Spectral reality and unitary time evolution are essential for any quantum theory. These features are guaranteed if H is Dirac Hermitian. (By Dirac Hermitian we mean that $H = H^\dagger$, where $\dagger$ represents combined complex conjugation and matrix transposition.) However, it is not necessary for H to be Dirac Hermitian for the spectrum to be real and for time evolution to be unitary; non-Dirac-Hermitian Hamiltonians can also define physically acceptable quantum theories.

The Hamiltonians (1) are $\mathcal{PT}$ symmetric because they are invariant under combined spatial reflection $\mathcal{P}$ and time reversal $\mathcal{T}$. Such Hamiltonians are physically acceptable because they are self-adjoint, not with respect to the Dirac adjoint $\dagger$, but rather with respect to $\mathcal{CPT}$ conjugation, where $\mathcal{C}$ is a linear operator that represents a hidden reflection symmetry of H. The $\mathcal{CPT}$ adjoint defines a positive-definite Hilbert space norm. Not every $\mathcal{PT}$-symmetric Hamiltonian has an entirely real spectrum, but the spectrum *is* entirely real if and only if a linear $\mathcal{PT}$-symmetric operator $\mathcal{C}$ exists that obeys three simultaneous algebraic equations [7]: $\mathcal{C}^2 = 1$, $[\mathcal{C}, \mathcal{PT}] = 0$, $[\mathcal{C}, H] = 0$. When the $\mathcal{C}$ operator exists, we say that the $\mathcal{PT}$ symmetry of H is unbroken. Finding the $\mathcal{C}$ operator is the crucial step in showing that time evolution for a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian is unitary. The phase transition between broken and unbroken regions for some $\mathcal{PT}$-symmetric Hamiltonians has been observed experimentally [3,4].

The Hamiltonians in (1) are smooth extensions in the parameter ϵ of the Dirac-Hermitian harmonic oscillator Hamiltonian (at $\epsilon = 0$) into the complex non-Hermitian domain ($\epsilon > 0$). As ϵ increases from 0, the Stokes' wedges in the complex-x plane inside of which the boundary conditions for the eigenvalue problem

$$-\psi''(x) + x^2(ix)^\epsilon \psi(x) = E\psi(x) \tag{2}$$

are imposed, rotate downward, and become thinner. As shown in Ref. [5], at $\epsilon = 0$ the Stokes' wedges are centered about the positive- and negative-real axes and have angular opening 90°. At $\epsilon = 2$ the Stokes' wedges are adjacent to and below the real axes and have angular opening 60°. When $\epsilon > 2$, these wedges lie below the real axis.

To illustrate the idea of this Letter in a quantum-mechanical context we set $\epsilon = 4$ in (1). The resulting x^6 Hamiltonian describes two different quantum theories because the eigenfunctions $\psi(x)$ can satisfy two different sets of boundary conditions [8]: (i) the conventional Dirac-Hermitian quantum theory for which $\psi(x)$ vanishes as $|x| \to \infty$ in the complex-x plane in 45° wedges centered about the real axes; or (ii) the unconventional $\mathcal{PT}$ theory, which is the extension in ϵ of the harmonic oscillator. For this non-Hermitian quantum theory $\psi(x)$ also vanishes as $|x| \to \infty$ in the complex plane in 45° wedges, but now these wedges are centered about $\arg x = -45°$ and $\arg x = -135°$. The one-point Green's function $G_1 = \langle x \rangle$ distinguishes between these two theories. The conventional Dirac-Hermitian theory has parity symmetry, and thus G_1 vanishes. The boundary conditions for the $\mathcal{PT}$ quantum theory violate parity symmetry, and as a result G_1 has a negative-imaginary value. The nonvanishing of G_1 in the $\mathcal{PT}$ theory is a purely nonperturbative effect; one cannot express G_1 for the Hamiltonian $H = p^2 + x^2 + gx^6$ as a series in powers of g.

The idea that different boundary conditions allow one Hamiltonian (or Lagrangian) to describe several different physical theories is general and extends beyond quantum mechanics to quantum field theories of fermion and/or boson fields of any spin and in any space-time dimension. However, for brevity we consider here the massless D-dimensional pseudoscalar field theory ($D < 2$) having a self-interaction of the form ϕ^{4n+2} ($n = 1, 2, 3, \ldots$). (Under parity reflection $\phi \to -\phi$.) The Euclidean functional integral for the vacuum persistence functional in the presence of an external source J is

$$Z[J] = \langle 0|0\rangle = \int_C \mathcal{D}\phi e^{-S},$$
$$S = \int d^D s \left[\frac{1}{2}(\nabla\phi)^2 + \frac{g}{4n+2}\phi^{4n+2} - J\phi\right]. \tag{3}$$

At $D = 1$, this quantum field theory reduces to a quasi-exactly solvable quantum-mechanical theory [9].

For each integer n there are $n + 1$ different physical realizations of the quantum field theory in (3). To explain this we consider the analogous one-dimensional integral $\int_C d\varphi \exp(-\varphi^{4n+2})$. When $n = 0$ this integral exists only if the integration contour C begins and ends in the Stokes' wedges of angular opening 90° centered about the real-φ axis. These Stokes' wedges are shown in Fig. 1. The contour C must begin and end in different Stokes' wedges; if C begins and ends in the same Stokes' wedge, the integral vanishes. When $n = 1$, there are *two* possible choices for integration contour C; C may connect the two 30°-Stokes' wedges centered about the real axis or C may connect the 30°-Stokes' wedge centered about −120° to the 30°-Stokes' wedge centered about −60° (see Fig. 2).

The contour C for $\int_C d\varphi \exp(-\varphi^6)$ must join a pair of $\mathcal{PT}$-symmetric Stokes' wedges (wedges that are symmetric about the imaginary axis) or else the integral is not real. A third pair of 30°-Stokes' wedges, one centered about 60° and the other centered about 120°, is not shown in Fig. 2;

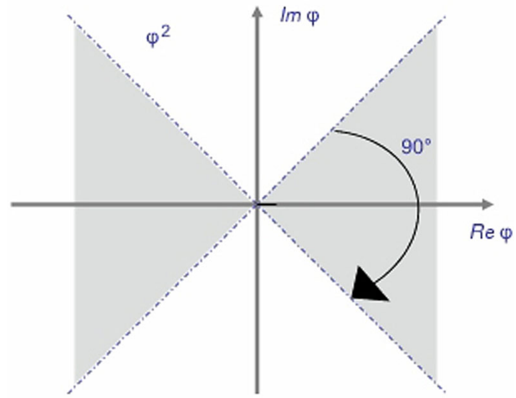

FIG. 1 (color online). Stokes' wedges (shaded regions) in the complex-φ plane in which the integration contour C for the integral $\int_C d\varphi \exp(-\varphi^2)$ terminates. This integral does not exist if C terminates in an unshaded wedge.

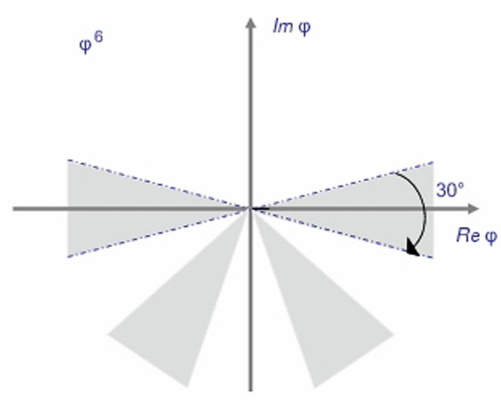

FIG. 2 (color online). Stokes' wedges (shaded regions) of angular opening 30° in which the integration contour C for the integral $\int_C d\varphi \exp(-\varphi^6)$ may terminate. The integral has two possible real values, one for which the contour connects the pair of wedges centered about the real axis and the other for which the contour connects the lower pair of wedges.

the integral exists if the contour C connects this pair of Stokes' wedges, but this case is not new; it is just the complex conjugate of the configuration in which C connects the $-120°$ and $-60°$ wedges.

The cases $n = 2$ (three pairs of 18° Stokes' wedges) and $n = 3$ (four pairs of 12.8° Stokes' wedges) are shown in Fig. 3. In the former case the $\int_C d\varphi \exp(-\varphi^{4n+2})$ has three independent real values; in the latter case it has four independent real values.

Returning to the quantum field theory with vacuum persistence function given in (3), we vary the action in the exponent and obtain the Euclidean field equation in the presence of the external c-number source $J(x)$:

$$-\nabla^2\phi(x) + g[\phi(x)]^{4n+1} = J(x). \quad (4)$$

This field equation is unique; it does not depend on the choice of complex contour C.

The expectation value of (4) in the vacuum state is

$$-\nabla^2 G_1(x) + g\langle[\phi(x)]^{4n+1}\rangle/Z[J] = J(x), \quad (5)$$

where $G_1(x)$ is the connected one-point Green's function:

$$G_1(x) = \frac{\delta \ln Z[J]}{\delta J(x)} = \frac{\langle\phi(x)\rangle}{Z[J]} = \int_C \mathcal{D}\phi\,\phi(x)e^{-S}. \quad (6)$$

This expectation value depends on the choice of metric, but in Ref. [10] it is shown that the path integral automatically gives the expectation value with the appropriate metric. Thus, if the integration contour terminates in the wedges containing the real axis, then the metric uses the conventional Dirac adjoint †, and if the contour terminates in another pair of Stokes' wedges, then the metric uses the $\mathcal{CPT}$ adjoint of the corresponding non-Dirac-Hermitian $\mathcal{PT}$-symmetric field theory [11].

To derive the Dyson-Schwinger equations for the connected Green's functions of the quantum field theory, we express the second term on the left side of (5) in terms of the higher connected Green's functions. The technique is standard (see, for example, Ref. [12]); one differentiates repeatedly with respect to the external source $J(x)$ and uses the formula for the n-point Green's function in the presence of the external source J:

$$G_n(x, y, z, \ldots) \equiv \delta^n/[\delta J(x)\delta J(y)\delta J(z)\cdots]\ln Z[J]. \quad (7)$$

We must truncate the Dyson-Schwinger equations in order to obtain a closed system. We consider here just the first two equations and neglect contributions from Green's functions beyond $G_2(x, y)$. This truncation gives the *mean-field* (or *one-pole*) approximation to the two-point Green's function. (Including higher Green's functions does not change any qualitative conclusions of this Letter.) Thus, we repeatedly differentiate with respect to $J(x)$ and get the following sequence of equations:

$$\begin{aligned}
&\langle 1\rangle = Z[J], \qquad \langle\phi(x)\rangle = G_1(x)Z[J],\\
&\langle[\phi(x)]^2\rangle = ([G_1(x)]^2 + G_2(x,x))Z[J],\\
&\langle[\phi(x)]^3\rangle = ([G_1(x)]^3 + 3G_1(x)G_2(x,x))Z[J],\\
&\langle[\phi(x)]^4\rangle = ([G_1(x)]^4 + 6[G_1(x)]^2G_2(x,x)\\
&\qquad + 3[G_2(x,x)]^2)Z[J],\\
&\langle[\phi(x)]^5\rangle = ([G_1(x)]^5 + 10[G_1(x)]^3G_2(x,x)\\
&\qquad + 15G_1[G_2(x,x)]^2)Z[J].
\end{aligned} \quad (8)$$

These expressions have a simple form as polynomials $P_n(t)$ in the variable $t = G_1(x)/\sqrt{G_2(x,x)}$,

$$\langle[\phi(x)]^n\rangle = [G_2(x,x)]^{n/2}Z[J]P_n(t), \quad (9)$$

where $P_n(t) = (-i)^n\mathrm{He}_n(it)$ are Hermite polynomials of imaginary argument: $P_0(t) = 1$, $P_1(t) = t$, $P_2(t) = t^2 + 1$, $P_3(t) = t^3 + 3t$, $P_4(t) = t^4 + 6t^2 + 3$, $P_5(t) = t^5 + 10t^3 + 15t$.

Next, we insert (9) into (5) and obtain

$$-\nabla^2 G_1(x) - i[G_2(x,x)]^{2n+1/2}\mathrm{He}_{4n+1}(it) = J(x). \quad (10)$$

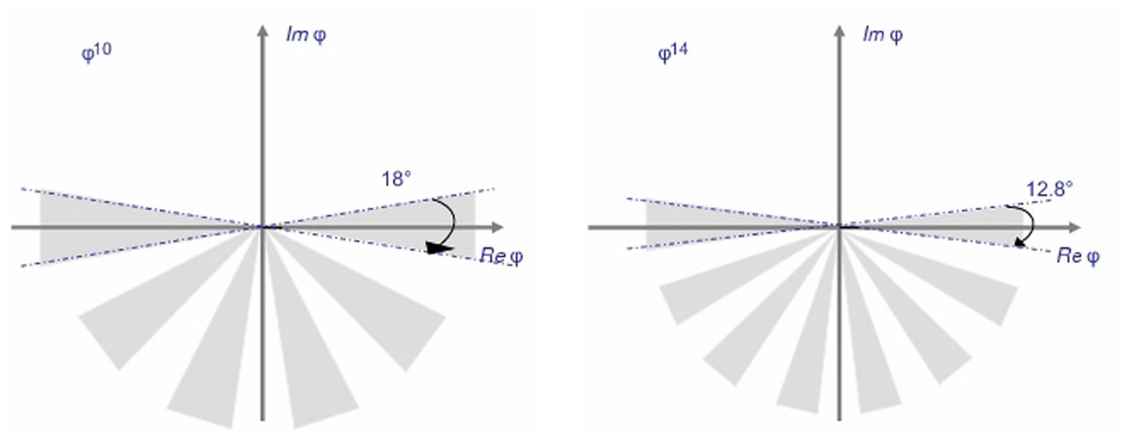

FIG. 3 (color online). Stokes' wedges (shaded regions) in which the integration contour C for $\int_C d\varphi \exp(-\varphi^{4n+2})$ may terminate. When $n = 2$ (left figure) there are three pairs of Stokes' wedges and when $n = 4$ there are four pairs of Stokes' wedges.

At $J \equiv 0$ translation invariance is restored, and $G_1(x)$ and $G_2(x, x)$ become the numbers G_1 and $G_2(0)$. Thus, the first of the truncated Dyson-Schwinger equations is

$$\mathrm{He}_{4n+1}[iG_1/\sqrt{G_2(0)}] = 0. \tag{11}$$

Note that the argument of He_{4n+1} remains invariant if wave-function renormalization is performed.

To obtain the second Dyson-Schwinger equation we differentiate (10) with respect to $J(y)$ and set $J \equiv 0$:

$$(-\nabla^2 + M^2)G_2(x - y) = \delta^D(x - y), \tag{12}$$

where the renormalized mass is given by

$$M^2 = [G_2(0)]^{2n}\mathrm{He}'_{4n+1}[iG_1/\sqrt{G_2(0)}]. \tag{13}$$

We solve (11)–(13) simultaneously: First, we Fourier transform (12) and find that in D-dimensional Euclidean space $\tilde{G}_2(p) = 1/(p^2 + M^2)$. Thus, for $0 \leq D < 2$ we get the finite result $G_2(0) = M^{D-2}\Gamma(1 - D/2)2^{-D}\pi^{-D/2}$. Second, we note that the Hermite polynomial He_{4n+1} is odd and only has real roots. There are two cases: either (i) $G_1 = 0$, which is the conventional Dirac-Hermitian parity-invariant solution to the Dyson-Schwinger equations, or (ii) we get $4n$ new parity-violating nonzero values for the one-point Green's function:

$$G_{1,j} = \pm iM^{-1+D/2}\sqrt{\Gamma(1 - D/2)}2^{-D/2}\pi^{-D/4}r_j, \tag{14}$$

where the dimensionless number r_j ($j = 1, \ldots 2n$) is one of the $2n$ positive roots of He_{4n+1}. Finally, we use the identity $\mathrm{He}'_{4n+1} = (4n + 1)\mathrm{He}_{4n}$ in (13) and use the interlacing-of-zeros property of the Hermite polynomials to prove that there are exactly n new positive values of M^2 corresponding to the nonzero values of $G_{1,j}$. This demonstrates the connection between pairs of Stokes' wedges and solutions to the Dyson-Schwinger equations.

For example, when $D = 1$ in a ϕ^6 model, $r_0 = 0$ and $r_1 = 2.856\,97$, and corresponding to these roots the dimensionless renormalized masses are $M = 1.391\,58$ and $M = 2.253\,99$. Thus, there are two families of particles: One particle (associated with a nonvanishing $G_{1,1}$) has a mass 1.62 times larger than that of the other particle (associated with a vanishing $G_{1,0}$). This ratio increases rapidly as a function of the space-time dimension D; for example, for $D = 0.0$, 0.5, 1.0, 1.5, 2.0, 2.5 this ratio takes the values 1.38, 1.47, 1.62, 1.90, 2.62, 6.88.

To conclude, while a flavor symmetry group is conventionally introduced to describe families of particles, we have shown that such families can arise naturally from the monodromy structure in the complex-field plane associated with rotation from one Stokes' wedge to another.

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