Quantum versus Classical Correlations in Gaussian States

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Quantum discord, a measure of genuinely quantum correlations, is generalized to continuous variable systems. For all two-mode Gaussian states, we calculate analytically the quantum discord and a related measure of classical correlations, solving an optimization over all Gaussian measurements. Almost all two-mode Gaussian states are shown to have quantum correlations, while for separable states, the discord is smaller than unity. For a given amount of entanglement, it admits tight upper and lower bounds. Via a duality between entanglement and classical correlations, we derive a closed formula for the Gaussian entanglement of formation of a family of three-mode Gaussian states.

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Entanglement, nonclassicality, and nonlocality are among the pivotal features of the quantum world. While for pure quantum states these concepts are like three facets of the same gemstone, they correspond to different resources in the general case of mixed states. Namely, while entanglement plays a central role in quantum communication [1], its necessity for mixed-state quantum computation is still unclear [2]. Conversely, several recent studies have shown that separable (that is, not entangled) states, traditionally referred to as "classically correlated," might retain some signatures of quantumness with potential operational applications for quantum technology [3–6]. One such signature is the quantum discord [7], which strives at capturing all the quantum correlations in a bipartite state, including—but not restricted to—entanglement. Significant progress in quantum information theory and implementation of quantum protocols has been recorded for both qubits and continuous variable systems [8]. However, there persists a fundamental gap between finite- and infinite-dimensional systems concerning the investigation of more general measures of quantumness versus classicality [9]. For Gaussian states, the workhorses of continuous variable quantum information, such an investigation would be especially valuable, since in view of the positivity of their Wigner distribution, these states have sometimes been tagged as essentially classical.

In this Letter we endeavor to bridge this gap. We define the quantum discord for Gaussian states and explicitly solve the optimization problem involved in its definition, constrained to measurements that preserve the Gaussian character of the states. We derive a closed formula for the ensuing Gaussian quantum discord and for a related measure of classical correlations [10] on all two-mode Gaussian states. We prove that these quantum correlations are limited for separable Gaussian states, yet they are nonzero for all but product states. For entangled states, quantum discord admits tight upper and lower bounds, given by functions of the Gaussian entanglement of formation [11]. Exploiting a duality between entanglement and classical correlations [12], we calculate exactly the Gaussian entanglement of formation of a class of mixed three-mode states. Our results unveil the general structure and nature of bosonic Gaussian correlations.

Quantum discord [7] originates from the discrepancy between two classically equivalent definitions of mutual information, a measure of total correlations in a quantum state. For classical probability distributions, the quantities $I(A{:}B) = H(A) + H(B) - H(A,B)$, $J(A{:}B) = H(A) - H(A|B)$, and $J(B{:}A) = H(B) - H(B|A)$ all coincide due to Bayes' rule, where H is the Shannon entropy and the conditional entropy $H(A|B)$ is an average of the Shannon entropies of A conditioned on the alternatives of B. For a bipartite quantum state ϱ_{AB}, the mutual information can be defined as $I(\varrho_{AB}) = S(\varrho_A) + S(\varrho_B) - S(\varrho_{AB})$, where S stands for the von Neumann entropy, $S(\varrho) = -\mathrm{Tr}(\varrho \log \varrho)$ (throughout the paper, log denotes the natural logarithm). The quantum analogue of $J(A{:}B)$, known as a one-way classical correlation and denoted as $\mathcal{J}^{\leftarrow}(\varrho_{AB})$, is operationally associated with the distillable common randomness between the two parties [13], and depends on the measurements $\{\Pi_i\}$, $\sum_i \Pi_i = \mathbb{1}$, made on B [10]. The state of A after the measurement is given by $\varrho_{A|i} = \mathrm{Tr}_B(\varrho_{AB}\Pi_i)/p_i$, $p_i = \mathrm{Tr}_{A,B}(\varrho_{AB}\Pi_i)$. A quantum analogue of the conditional entropy can then be defined as $\mathcal{H}_{\{\Pi_i\}}(A|B) \equiv \sum_i p_i S(\varrho_{A|i})$, and the one-way classical correlation, maximized over all possible measurements, takes the form $\mathcal{J}^{\leftarrow}(\varrho_{AB}) = S(\varrho_A) - \inf_{\{\Pi_i\}} \mathcal{H}_{\{\Pi_i\}}(A|B)$. The quantum discord is finally defined as total minus classical correlations:

$$\begin{aligned}\mathcal{D}^{\leftarrow}(\varrho_{AB}) &= I(\varrho_{AB}) - \mathcal{J}^{\leftarrow}(\varrho_{AB}) \\ &= S(\varrho_B) - S(\varrho_{AB}) + \inf_{\{\Pi_i\}} \mathcal{H}_{\{\Pi_i\}}(A|B). \end{aligned} \tag{1}$$

We denote by $\mathcal{J}^{\rightarrow}(\varrho_{AB})$ and $\mathcal{D}^{\rightarrow}(\varrho_{AB})$ the corresponding (generally different) quantities, where the roles of A and B are swapped. On pure states, quantum discord coincides with the entropy of entanglement $S(\varrho_B) = S(\varrho_A)$. States with zero discord represent essentially a classical probability distribution embedded in a quantum system, while a positive discord, even on separable (mixed) states, is an indicator of quantumness that arises, e.g., when ϱ_{AB} has entangled eigenvectors [6,14], and may operationally be associated with the impossibility of local broadcasting [4].

We now define a Gaussian version of quantum discord and calculate it analytically for all two-mode Gaussian states. A two-mode Gaussian state ϱ_{AB} is fully specified, up to local displacements, by its covariance matrix (CM) $\boldsymbol{\sigma}_{AB}$ of elements $\sigma_{ij} = \mathrm{Tr}[\varrho_{AB}\{\hat{R}_i, \hat{R}_j\}_+]$, where $\hat{\mathbf{R}} = (\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B)$ is the vector of phase-space operators [9]. By means of local unitary (symplectic at the CM level) operations, every two-mode CM can be transformed in a standard form with diagonal sub-blocks,

$$\boldsymbol{\sigma}_{AB} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \boldsymbol{\gamma}^T & \boldsymbol{\beta} \end{pmatrix}, \tag{2}$$

with $\boldsymbol{\alpha} = a\mathbb{1}$, $\boldsymbol{\beta} = b\mathbb{1}$, and $\boldsymbol{\gamma} = \mathrm{diag}\{c, d\}$. Let us define the symplectic invariants $A = \det\boldsymbol{\alpha}$, $B = \det\boldsymbol{\beta}$, $C = \det\boldsymbol{\gamma}$, and $D = \det\boldsymbol{\sigma}_{AB}$. The CM corresponds to a physical state iff $A, B \geq 1$ and $\nu_\pm \geq 1$, where the symplectic eigenvalues are defined by $2\nu_\pm^2 = \Delta \pm \sqrt{\Delta^2 - 4D}$ with $\Delta = A + B + 2C$. A Gaussian state with CM $\boldsymbol{\sigma}_{AB}$ is entangled iff $\tilde{\nu}_- < 1$, where the smallest symplectic eigenvalue $\tilde{\nu}_-$ of the partially transposed CM is obtained from ν_- by replacing C with $-C$ (i.e., by time reversal) [15].

Both $\mathcal{J}^\leftarrow$ and $\mathcal{D}^\leftarrow$ are invariant under local unitaries [7,10]. Hence, we can derive their closed formulas by exploiting the standard form of a general two-mode Gaussian state, and we can later recast our results in terms of the four invariants. The Gaussian quantum discord of a two-mode Gaussian state ϱ_{AB} can be defined as the quantum discord where the conditional entropy is restricted to generalized Gaussian positive operator valued measurements (POVMs) on B. These measurements are all executable using linear optics and homodyne detection [16]. We then have $\mathcal{D}^\leftarrow(\varrho_{AB}) = S(\varrho_B) - S(\varrho_{AB}) + \inf_{\Pi_B(\eta)} \int d\eta\, p_B(\eta) S(\varrho_{A_\eta})$. Here the Gaussian measurement $\Pi_B(\eta)$ on subsystem B can be written, in general, as $\Pi_B(\eta) = \pi^{-1}\hat{W}_B(\eta)\Pi_B^0\hat{W}_B^\dagger(\eta)$, where $\hat{W}_B(\eta) = \exp(\eta\hat{b}^\dagger - \eta^*\hat{b})$ is the Weyl operator, $\hat{b} = (\hat{x}_B + i\hat{p}_B)/\sqrt{2}$, $\pi^{-1}\int d^2\eta\,\Pi_B(\eta) = \mathbb{1}$, and Π_B^0 is the density matrix of a (generally mixed) single-mode Gaussian state. The conditional entropy is a concave function of the POVM elements; i.e., it is concave on the set of single-mode Gaussian states Π_B^0. Gaussian states do not form a convex set, yet every Gaussian state admits a convex decomposition into pure Gaussian states. Thus, it is sufficient (as in the finite-dimensional case) to restrict to states Π_B^0 that are pure, single-mode Gaussian states [17] whose CM we denote as $\boldsymbol{\sigma}_0$. The conditional state $\varrho_{A|\eta}$ of subsystem A after the measurement $\Pi_B(\eta)$ on B has a CM independent of the measurement outcome [16] and given by $\boldsymbol{\varepsilon} = \boldsymbol{\alpha} - \boldsymbol{\gamma}(\boldsymbol{\beta} + \boldsymbol{\sigma}_0)^{-1}\boldsymbol{\gamma}^T$. Recalling then that the von Neumann entropy of an n-mode Gaussian state with CM $\boldsymbol{\sigma}$ can be computed as [18] $S(\boldsymbol{\sigma}) = \sum_{i=1}^N f(\nu_i)$, where ν_i are the symplectic eigenvalues of the state and $f(x) = (\frac{x+1}{2})\log[\frac{x+1}{2}] - (\frac{x-1}{2})\log[\frac{x-1}{2}]$, the one-way classical correlation and the Gaussian quantum discord for two-mode Gaussian states with CM $\boldsymbol{\sigma}_{AB}$ are

$$\mathcal{D}^\leftarrow(\sigma_{AB}) = f(\sqrt{B}) - f(\nu_-) - f(\nu_+) + \inf_{\boldsymbol{\sigma}_0} f(\sqrt{\det\boldsymbol{\varepsilon}}),$$
$$\mathcal{J}^\leftarrow(\sigma_{AB}) = f(\sqrt{A}) - \inf_{\boldsymbol{\sigma}_0} f(\sqrt{\det\boldsymbol{\varepsilon}}). \tag{3}$$

To get closed formulas we need to minimize $\det(\boldsymbol{\varepsilon})$ over all CMs $\boldsymbol{\sigma}_0$ corresponding to pure one-mode Gaussian states, i.e., rotated squeezed states: $\boldsymbol{\sigma}_0 = R(\theta)\mathrm{diag}\{\lambda, 1/\lambda\}R^T(\theta)$, where $\lambda \geq 0$ and $R(\theta) = \{[\cos\theta, \sin\theta], [-\sin\theta, \cos\theta]\}$. For a general two-mode Gaussian state in standard form, one has $E(\lambda, \theta) \doteq \det(\boldsymbol{\varepsilon}) = [2a^2(b+\lambda)(1+b\lambda) - a(c^2 + d^2)(2b\lambda + \lambda^2 + 1) + a(c^2 - d^2)(\lambda^2 - 1)\cos(2\theta) + 2c^2d^2\lambda]/[2(b+\lambda)(1+b\lambda)]$. We set, without loss of generality, $c \geq |d|$. We now look for stationary points of E by studying its partial derivatives. One such point is $\lambda = 1$, for which $\boldsymbol{\sigma}_0$ is the identity (regardless of θ) and is a saddle point except when $d = \pm c$. Next, the equation $\partial_\lambda E = 0$ is quadratic in λ, but one of its roots is always negative. The other root, given by $\lambda = \lambda_2 = [ab(d^2 - c^2) + c|d|\sqrt{(a - ab^2 + bc^2)(a - ab^2 + bd^2)}]/[ab^2c^2 - (a + bc^2)d^2]$ and $\theta = 0$ (or, equivalently, $1/\lambda_2$ and $\theta = \pi/2$), is a local minimum and is acceptable provided that $\lambda_2 \geq 0$, that is, when $\delta = [-ac_1^2 + b(ab - c_1^2)c_2^2] \geq 0$. Additional candidates for $\inf E$ have to be sought at the boundaries of the parameter space: a potential minimum lies at $\lambda \to 0$, $\theta = 0$ (or, equivalently, $\lambda \to \infty$, $\theta = \pi/2$). In the whole physically allowed region for the CM parameters a, b, c, and d, we have $E(1, 0) \geq E(\lambda_2, 0)$, with equality holding only when $d = \pm c$, and $E(\lambda_2, 0) \leq E(0, 0)$. Thus, for any $\boldsymbol{\sigma}_{AB}$, $\inf_{\boldsymbol{\sigma}_0}\det(\boldsymbol{\varepsilon})$ is equal to $E(\lambda_2, 0)$ if $\delta \geq 0$ and to $E(0, 0)$ otherwise. In terms of symplectic invariants, the Gaussian quantum discord and the one-way classical correlation for a general two-mode Gaussian state $\boldsymbol{\sigma}_{AB}$ are given by Eq. (3) with

$$E^{\min} = \inf_{\boldsymbol{\sigma}_0} \det(\boldsymbol{\varepsilon}) = \begin{cases} \frac{2C^2 + (-1+B)(-A+D) + 2|C|\sqrt{C^2 + (-1+B)(-A+D)}}{(-1+B)^2} & (D - AB)^2 \leq (1 + B)C^2(A + D); \\ \frac{AB - C^2 + D - \sqrt{C^4 + (-AB+D)^2 - 2C^2(AB+D)}}{2B} & \text{otherwise.} \end{cases} \tag{4}$$

For states falling in the second case of Eq. (4), homodyne measurements (projections onto infinitely squeezed states, $\lambda = 0$) on B minimize the conditional entropy of A. An example is when

$$A = D = a^2, \qquad C = (1 - B)/2, \qquad B = b^2, \tag{5}$$

with $1 \leq b \leq 2a - 1$, which is a mixed state of partial minimum uncertainty [9]. On the other hand, the first case corresponds to a more general measurement, i.e., projection of mode B onto a squeezed state with unbalanced, finite variances on $\hat{x}_B$ and $\hat{p}_B$. A notable class of states satisfying the first case are squeezed thermal states (including pure states), characterized by $d = \pm c$, for which the conditional entropy is, in particular, minimized by heterodyne measurements (projection onto coherent states, $\lambda = 1$). In general, Gaussian quantum discord can be accessed experimentally by linear optics.

We now analyze the relationships between classical correlations, quantum discord, separability, and entanglement. For every entangled state the quantum discord is strictly positive [since $S(\varrho_B) - S(\varrho_{AB}) > 0$]. Almost all separable states in finite dimensions also have nonzero discord [5]. In any dimension (including infinite dimensions under the constraint of finite mean energy), the states ϱ_{AB} with zero discord are the ones that saturate the strong subadditivity inequality for the von Neumann entropy on a tripartite state ϱ_{ABC}, where C is an ancillary system realizing the measurements on B [14,19]. From the characterization of such states in the Gaussian scenario [20] (see Appendix A in [21] for more details), it follows that the only two-mode Gaussian states with zero Gaussian quantum discord are product states $\boldsymbol{\sigma}_{AB} = \boldsymbol{\alpha} \oplus \boldsymbol{\beta}$, i.e., states with no correlations at all, that constitute a zero measure set. Quite remarkably, then, all correlated two-mode Gaussian states have nonclassical correlations certified by a nonzero quantum discord.

For Gaussian states with asymptotically diverging mean energy, however, interesting correlation structures arise. Consider the squeezed thermal state given by

$$a = \cosh(2s), \qquad b = \cosh^2 r \cosh(2s) + \sinh^2 r, \qquad c = -d = \cosh r \sinh(2s). \tag{6}$$

For $r = 0$, this is a pure two-mode squeezed vacuum state, whose entanglement is an increasing function of s. In the limit $r \to \infty$, it is asymptotically separable (but not in product form). Concerning the discord (minimized in this example by heterodyne detections), we find $\mathcal{D}^{\leftarrow}(\boldsymbol{\sigma}_{AB}) = f[\cosh^2 r \cosh(2s) + \sinh^2 r] - f[\cosh^2 r + \cosh(2s)\sinh^2 r] \overset{r,s\to\infty}{\longrightarrow} 0$ and $\mathcal{D}^{\rightarrow}(\boldsymbol{\sigma}_{AB}) = f[\cosh(2s)] - f[\cosh^2 r + \cosh(2s)\sinh^2 r] + f[\cosh(2r)] \overset{r,s\to\infty}{\longrightarrow} 1$. While these limiting values are associated with ideal, unnormalizable states, they can be approached arbitrarily close by physical Gaussian states with large, but finite mean energy. Hence, surprisingly, there exist bipartite Gaussian states such that (i) they are nonproduct states, with an arbitrarily large correlation matrix $\boldsymbol{\gamma}$, yet have infinitesimal quantum discord, and (ii) their quantum correlations can be revealed by probing only one subsystem, but not the other. Thus motivated, we have explored the discord asymmetry for 1×10^6 randomly generated (separable and entangled) two-mode Gaussian states. Let $\mathcal{D}^{\max} = \max\{\mathcal{D}^{\leftarrow}, \mathcal{D}^{\rightarrow}\}$, $\mathcal{D}^{\min} = \min\{\mathcal{D}^{\leftarrow}, \mathcal{D}^{\rightarrow}\}$ for a given CM. We find numerically that $\mathcal{D}^{\max} - \mathcal{D}^{\min} \leq \mathcal{D}^{\min}/[\exp(\mathcal{D}^{\min}) - 1] \leq 1$. The leftmost bound is saturated by states of Eq. (6) in the limit $s \to \infty$, and unity is reached for $r \to \infty$ as well. The maximum discord asymmetry decays exponentially with $\mathcal{D}^{\min}$, so when the discord calculated either way is large, we have *de facto* $\mathcal{D}^{\leftarrow} = \mathcal{D}^{\rightarrow}$.

Next we ask the following question: To what extent can separable Gaussian states be quantumly correlated? While their discord is typically nonzero (except for product states), we find that it cannot exceed one unit of information [Fig. 1 (left panel)]. In Appendix B in [21] we prove that for all two-mode separable Gaussian states, $\mathcal{D}^{\leftarrow}(\boldsymbol{\sigma}_{AB}^{\text{sep}}) \leq [(b-1)/2]\log[(b+1)/(b-1)] \leq 1$. The first inequality is saturated by separable squeezed thermal states whose correlation matrix has a maximum determinant C and whose CM has maximum asymmetry between the two modes: $c = d = 1 + ab - a - b$, $a \to \infty$ [solid (red) curve in Fig. 1 (left panel)]. The second bound is reached for $b \to \infty$. This implies a sufficient condition for the entanglement of Gaussian states given their discord: if $\mathcal{D}^{\leftarrow}(\boldsymbol{\sigma}_{AB}) > 1$, then $\boldsymbol{\sigma}_{AB}$ is entangled.

We now focus on entangled states, and study how Gaussian quantum discord compares quantitatively to the entanglement of the states, specifically measured by the most "compatible" measure available, the Gaussian entanglement of formation (Gaussian EoF) $\mathcal{E}_G$ [11]. This is defined for Gaussian states ϱ_{AB} as the convex roof of the von Neumann entropy of entanglement, restricted to decompositions of ϱ_{AB} into pure Gaussian states. This can be evaluated via a minimization over CMs: $\mathcal{E}_G(\boldsymbol{\sigma}_{AB}) = \inf_{\boldsymbol{\sigma}'_{AB} \leq \boldsymbol{\sigma}_{AB}:\ \det(\boldsymbol{\sigma}'_{AB})=1} f(\sqrt{\det \boldsymbol{\alpha}'})$, where the infimum runs over all pure bipartite Gaussian states with CM $\boldsymbol{\sigma}'_{AB}$ smaller than $\boldsymbol{\sigma}_{AB}$, and $\boldsymbol{\alpha}'$ is the reduction of $\boldsymbol{\sigma}'_{AB}$ corresponding to the marginal state of mode A. Compact formulas for $\mathcal{E}_G$ exist for all symmetric two-mode states (where the Gaussian EoF coincides with the true EoF, as the Gaussian decomposition is optimal) [22], as well as asymmetric ones [11,23]. In Fig. 1 (right panel), we plot $\mathcal{D}^{\leftarrow}$ vs $\mathcal{E}_G$ for 30 000 randomly generated two-mode Gaussian states. We find that for a given entanglement degree, the discord is bounded both from above and below. To find the upper bound analytically, we can restrict, as in the separable case, to squeezed thermal states (see [21]) with $d = -c$. Further optimization within this family of states yields that, for all two-mode entangled Gaussian states, the quantum discord satisfies

$$\mathcal{D}^{\leftarrow}(\boldsymbol{\sigma}_{AB}) \leq \max\{\mathcal{E}_G(\boldsymbol{\sigma}_{AB}), 2\cosh^2 r \log(\coth r)\}, \tag{7}$$

where $\mathcal{E}_G(\boldsymbol{\sigma}_{AB}) = f(1 + 2\sinh^{-2} r)$ implicitly defines r. The rightmost bound [solid (red) curve in Fig. 1 (right panel)] dominates in the low entanglement regime ($\mathcal{E}_G < 2\log 2$) and corresponds to $\mathcal{D}^{\rightarrow}$ of the states of Eq. (6) in the limit $s \to \infty$. The leftmost bound in Eq. (7) [dotted (green) line in Fig. 1 (right panel)] is instead reached on pure states, and sets an upper limit to the quantum discord of all two-mode

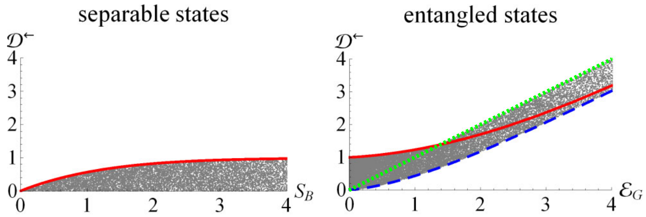

FIG. 1 (color online). Left panel: Gaussian quantum discord versus marginal entropy for separable two-mode Gaussian states. Right panel: Gaussian quantum discord versus Gaussian EoF for entangled two-mode Gaussian states. See text for details of the bounding curves.

Gaussian states with sufficiently high entanglement. On the other hand, for a given $\mathcal{E}_G$, the Gaussian discord also satisfies $\mathcal{D}^{\leftarrow}(\boldsymbol{\sigma}_{AB}) \geq 2\log(\coth r)$, with r as before. This lower bound follows from the fact that states of Eq. (6) with $s \rightarrow \infty$ are extremal for the discord asymmetry, and corresponds to $\mathcal{D}^{\leftarrow}$ of those states [dashed (blue) line in Fig. 1 (right panel)]. Interestingly this entails that, asymptotically, for all two-mode Gaussian states with $\mathcal{E}_G \gg 0$, their discord lies between $\mathcal{E}_G - 1$ and $\mathcal{E}_G$.

A further key result of our study is that Eq. (4) provides a closed, computable formula for the Gaussian EoF of a class of three-mode mixed Gaussian states. This is possible thanks to a duality relation between (Gaussian) classical correlations and (Gaussian) EoF [12]. Let ϱ_{ABST} be a purification of ϱ_{AB}, i.e., a pure (Gaussian) state such that $\mathrm{Tr}_{ST}[\varrho_{ABST}] = \varrho_{AB}$ (we need, in general, two ancillary modes S and T to construct such a purification [18]). Then, $\mathcal{J}^{\leftarrow}(\varrho_{AB}) + \mathcal{E}(\varrho_{AST}) = S(\varrho_A)$, where $\mathcal{E}(\varrho_{AST})$ denotes the EoF between party A and the block of modes ST. In the Gaussian framework, from Eq. (3) we have simply $\mathcal{E}_G(\boldsymbol{\sigma}_{AST}) = \inf_{\boldsymbol{\sigma}_0} f(\sqrt{\det \boldsymbol{\varepsilon}})$. The states with CM $\boldsymbol{\sigma}_{AST}$ encompass all three-mode Gaussian states that are reductions of a four-mode pure Gaussian state. Their symplectic spectrum is of the form $\{1, 1, b\}$; i.e., they are mixed states of partial minimum uncertainty, with two vacua as normal modes. For all such states, we now present an analytic method to compute the Gaussian EoF across the bipartition $A \times (ST)$: first, construct a purification; i.e., append an ancillary mode B [with $\det(\boldsymbol{\sigma}_B) = \det(\boldsymbol{\sigma}_{AST}) = b^2$] such that $\boldsymbol{\sigma}_{ABST}$ is pure [11,18]. Then, evaluate $E^{\min}$ of the marginal state $\boldsymbol{\sigma}_{AB}$ from Eq. (4). Finally, $f(\sqrt{E^{\min}})$ is the Gaussian EoF between A and ST. An example of such a state, of relevance in a cryptographic setting, is provided in [21] (Appendix C).

For states $\boldsymbol{\sigma}_{AB}$ with $\nu_- = 1$, the purification requires a single ancillary mode S, and the Gaussian EoF between modes A and S, as computed through Eq. (4), agrees with the formula derived in [23]. In particular, for the states of Eq. (5), the complementary state of modes AS is symmetric; the findings of Ref. [22] thus imply that the Gaussian POVM devised here is globally optimal for the calculation of their (unconstrained) quantum discord. For general two-mode Gaussian states, it is an open question whether non-Gaussian measurements (e.g. photo-detection) can lead to a further minimization of the discord.

This Letter paves the way for the study of general quantum correlations in multimode harmonic lattices, and through the paradigmatic two-mode case, demonstrates the ''truly quantum'' nature of Gaussian states, reinforcing their key role in quantum information processing.

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Note added.—After completion of our study, another work appeared [24] where Gaussian quantum discord is independently defined, and explicitly computed only for two-mode squeezed thermal states.

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