

S-Duality Constraints on 1D Patterns Associated with Fractional Quantum Hall States

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(Received 23 February 2010; published 6 July 2010)

Using the modular invariance of the torus, constraints on the 1D patterns are derived that are associated with various fractional quantum Hall ground states, e.g., through the thin torus limit. In the simplest case, these constraints enforce the well-known odd-denominator rule, which is seen to be a necessary property of all 1D patterns associated to quantum Hall states with minimum torus degeneracy. However, the same constraints also have implications for the non-Abelian states possible within this framework. In simple cases, including the $\nu = 1$ Moore-Read state and the $\nu = 3/2$ level 3 Read-Rezayi state, the filling factor and the torus degeneracy uniquely specify the possible patterns, and thus all physical properties that are encoded in them. It is also shown that some states, such as the “strong p -wave pairing state,” cannot in principle be described through 1D patterns.

DOI: 10.1103/PhysRevLett.105.026802

PACS numbers: 73.43.-f

Introduction.—The study of fractional quantum Hall (FQH) liquids has been among the most intriguing problems in condensed matter physics during the past few decades, in both theory and experiment. On the theoretical side, the construction of variational many-body wave functions has traditionally played a pivotal role [1]. In principle, the possible variational constructions are limitless. A systematic classification of FQH phases therefore requires additional constraints, such as simplicity in a composite fermion picture [2]. Another program to implement such constraints is to require that the trial wave functions can be obtained as conformal blocks in certain conformal field theories (CFTs) [3]. The problem is then relegated to identifying all conformal field theories leading to permissible wave functions. On the other hand, it has recently become appreciated that a large class of trial wave functions can be characterized by simple sequences of integers, either through the thin torus limit and adiabatic continuity [4–9], or through Jack polynomials [10]. The patterns of integers associated with viable quantum Hall states are in turn subject to a number of consistency requirements, such as rotational invariance of the associated Jack polynomials, or constraints on the associated “patterns of zeros” studied in Ref. [11]. A complete set of consistency requirements is desirable in order to understand the possible quantum numbers of all quantum Hall phases that are accessible within this framework. In this Letter, it will be shown that the one-dimensional (1D) patterns associated with the ground state sectors of a quantum Hall phase are highly constrained by modular invariance on the torus. This allows one to more systematically identify patterns corresponding to quantum Hall states, which may be constructed explicitly through Jack polynomials once viable patterns are identified. Intimate connections between the latter and CFTs are currently being explored [12], such that one may hope that states thus constructed also fit within the paradigm based on CFTs. As it turns out, in the simplest cases the constraints derived here just imply

the well-known “odd denominator rule,” which is found to be required within this framework for all quantum Hall states that have the “minimum torus degeneracy”. Furthermore, in some other cases of interest, it is found that the filling factor and the torus degeneracy already completely determine the associated set of 1D patterns. These patterns, in turn, can be shown to have direct implications for the statistics of quasiparticles, using the method of Ref. [13]. This is, in particular, true for the Moore-Read state [3], where the statistics are fully determined modulo a certain Abelian phase [13]. Similar statements apply [14] to the $k = 3$ Read-Rezayi state [15]. It is thus found that within this framework, the filling factor and the torus degeneracy alone may greatly constrain the low energy physics in some cases. Furthermore, it will also be shown that the “strong p -wave pairing” state does not admit a description in terms of 1D patterns.

1D patterns and S duality.—The bulk properties of fractional quantum Hall phases are expected to be independent of the topology of the two-dimensional (2D) space they live in. Here the topology of choice will be the torus. As an example, let us consider how the patterns arise that are associated with the three degenerate ground states of the $\nu = 1$ (Moore-Read) Pfaffian, i.e., 20202020... , 02020202... and 11111111... [5]. A standard basis of the lowest Landau level (LL) is given by orbitals φ_n , which are localized around $x \approx nL_x/L$ along the x axis, where L_x , L_y are the dimensions of the torus, and $L = L_x L_y / 2\pi$ is the LL degeneracy. Moreover, the state φ_n is completely delocalized in y . We can expand the Pfaffian ground states in a basis of the LL Fock space:

$$|\text{GS}\rangle = \sum_{\{m_n\}} C_{\{m_n\}} |m_1, m_2 \dots m_L\rangle. \quad (1)$$

Here, m_n denotes the number of particles (bosons) in the state φ_n . The $C_{\{m_n\}}$ depend on the aspect ratio L_x/L_y . In the thin torus limit $L_y \rightarrow 0$, the Pfaffian ground states evolve adiabatically [5] into states dominated by a *single* pattern

of occupancy numbers $\{m_n\}$, corresponding to the states $|202020\dots\rangle$, $|020202\dots\rangle$, and $|111111\dots\rangle$, respectively. The patterns of occupancy numbers characterizing these limiting states still carry a wealth of information about the original “large torus” quantum Hall states, and, in particular, carry the same quantum numbers under magnetic translations. As pointed out initially, a very large and important class of quantum Hall states can be characterized through simple patterns in this way. Moreover, on the torus the roles of x and y can be exchanged through the modular S transformation, which leaves the physics invariant. Hence the same patterns can be obtained in the “dual” limit $L_x \rightarrow 0$. However, the limiting states will then not assume the simple occupancy patterns in the φ_n basis referred to in Eq. (1), but in a dual LL basis $\bar{\varphi}_n$. The $\bar{\varphi}_n$ can be thought of as a “rotated” version of the φ_n , and are related to the φ_n via Fourier transform (see, e.g., Ref. [16]).

A rather stringent constraint on legitimate thin torus patterns can be obtained by exploring the consequences of modular S invariance on the representation of the magnetic translation group formed by the ground states. The magnetic translation group is generated by operators T_x and T_y that act on single particle LL orbitals via

$$T_x \varphi_n(z) = \varphi_{n+1}(z) \quad T_y \varphi_n(z) = e^{-\frac{2\pi i}{L} n} \varphi_n(z) \quad (2a)$$

$$T_x \bar{\varphi}_n(z) = e^{\frac{2\pi i}{L} n} \bar{\varphi}_n(z) \quad T_y \bar{\varphi}_n(z) = \bar{\varphi}_{n+1}(z), \quad (2b)$$

cf., e.g., Ref. [16]. Here, L is the total number of LL orbitals. The torus ground states of a given quantum Hall phase form a representation of the magnetic translation group. This representation cannot depend on the aspect ratio of the torus. If a simple thin torus limit exists in the sense described above, it allows one to immediately infer the matrices $R(T_x)$, $R(T_y)$ of this representation. In the limit $L_y \rightarrow 0$, where the patterns extend along the x direction and correspond to simple product states in the φ_n basis, (2a) implies that such product states are eigenstates of T_y with eigenvalue $\exp(-2\pi i/L \sum_j n_j)$. Here, n_j is the orbital index of the j th particle in the pattern. Likewise, T_x performs a right shift of the pattern.

As mentioned above, modular S invariance implies that in the opposite thin torus limit, $L_x \rightarrow 0$, the *same* thin torus patterns must appear. These patterns now extend along y and correspond to simple product states in the $\bar{\varphi}_n$ basis. In general, however, a ground state that evolves into a given 1D pattern in one thin torus limit will evolve into a superposition of such patterns in the opposite thin torus limit, and vice versa. As a result, the representation matrices obtained from the product ground states (patterns) in the two mutually dual thin torus limits are unitarily equivalent, but not identical. Equation (2) immediately implies the following relations when passing from the $L_y \rightarrow 0$ limit to the dual limit $L_x \rightarrow 0$:

$$R(T_x) = \bar{R}(T_y), \quad R(T_y) = \bar{R}(T_x)^\dagger. \quad (3)$$

In the above, $R(T_{x,y})$ refers to the matrices describing T_x

and T_y in the basis of product ground states emerging in the $L_y \rightarrow 0$ limit. Let us label these states by $|\tau\rangle$, where τ denotes the associated simple pattern, e.g., $\tau = 2020\dots, 1111\dots, 0202\dots$ for the $\nu = 1$ Pfaffian. $\bar{R}(T_{x,y})$ are the matrices describing the transformation properties of the dual product states $|\bar{\tau}\rangle = \hat{S}|\tau\rangle$, where \hat{S} is the unitary transformation that takes the orbital φ_n into $\bar{\varphi}_n$. Equations (3) then immediately follow from Eqs. (2). Apparently, the state $|\tau\rangle$ in general has different transformation properties from its dual version $|\bar{\tau}\rangle$. Hence these states correspond to opposite thin torus limits of *different* degenerate ground states. On the other hand, so long as the ground state patterns describe the translational properties of the ground states for any aspect ratio of the torus (as implied, e.g., by adiabatic continuity), the matrices R and \bar{R} must form the same representation. They must therefore be related by a unitary transformation, $\bar{R}(T_{x,y}) = U^\dagger R(T_{x,y}) U$. Equation (3) then becomes:

$$R(T_x) = U^\dagger R(T_y) U, \quad R(T_y) = U^\dagger R(T_x)^\dagger U. \quad (4)$$

The above says that the representation of the magnetic translation group implied by the patterns must be “self-dual”. From given ground state patterns, it is always easy to work out these matrices from (2) as described above. Equation (4) then poses severe constraints on these patterns.

Odd denominator rule.—A nontrivial torus degeneracy is a hallmark of topologically ordered systems. It is well understood that quite generally, if the system is characterized by a filling factor $\nu = p/q$, with p and q coprime, its minimum torus degeneracy is q [17,18]. This lower bound is typically exceeded in time-reversal invariant systems. It is, however, satisfied for the simplest fractional quantum Hall states, such as those in the Abelian hierarchy. Hierarchy states are also known for their compliance with the “odd denominator rule,” according to which q must be odd. This has been understood in various ways [19,20]. Here it will be shown that for all states that can be represented through periodic 1D patterns as discussed, the odd denominator rule is a direct consequence of the minimum torus degeneracy, together with the requirement (4).

Suppose, now, an incompressible quantum Hall state can be represented by a 1D pattern with a unit cell containing p particles and q orbitals, Fig. 1, such that the LL filling factor is $\nu = p/q$. By translational symmetry, there must be at least q ground states on the torus, since evidently q

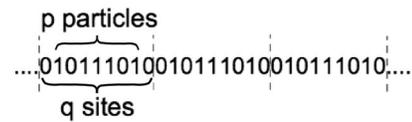


FIG. 1. A generic periodic 1D pattern as it might appear in the thin torus limit of a fractional quantum Hall state, for a given topological sector. In addition, the individual orbitals may carry a pseudospin label, as in Ref. [6]. The complete set of patterns for all topological sectors are subject to the duality constraint (4).

translations are required to transform the state back to itself. If we further assume that the state has the minimum torus degeneracy, it follows that *all* ground states are related by translation *and* that p and q are coprime. Instead, however, I will start from the weaker assumption that all ground states are related by translation. Using Eq. (4), this is already sufficient to show that p and q are coprime, and hence the state has the minimum torus degeneracy. To see this, let the state associated with the pattern be denoted by $|\tau\rangle$. The states $T_x^j|\tau\rangle$, $j = 0, \dots, q-1$ then represent a complete set of torus ground states. From these, we can easily form eigenstates of T_x with q distinct eigenvalues. In contrast, T_y is found to have q distinct eigenvalues only if p and q are coprime. Note that the states $T_x^j|\tau\rangle$ are already eigenstates of T_y with eigenvalues that can be read off directly from the associated patterns as described above. Each application of T_x changes the T_y eigenvalue by a factor $\exp(-2\pi i\nu)$. It follows from this that if $\nu = p/q = p'/q'$ where p' and q' are coprime, the state $|\tau\rangle$ and its translated counterparts have exactly q' distinct T_y eigenvalues. The S -duality requirement Eq. (4) then implies that $q = q'$ since T_x and T_y must, in particular, have the same spectrum within the ground state space. Hence also $p = p'$. Thus one finds that whenever all torus ground states of a specific quantum Hall phase are related by translation, any permissible 1D pattern associated with this phase must satisfy that the size q of its unit cell and the number p of particles contained therein are coprime. To proceed, let us further assume that the total number N of particles in the state is even. This can be done without any loss of generality, since when some incompressible quantum Hall fluid exists for odd particle number on the torus, it also exists for even particle number by means of doubling the system size. The operators T_x^q and T_y^q are constant within the space spanned by the q ground states, since $|\tau\rangle$ is an eigenstate of both, and both operators commute with T_x . By the duality constraint Eq. (4), both operators must therefore be equal. By acting on $|\tau\rangle$, one easily finds that $T_x^q = (-1)^{fp}$, where $f = 1$ for fermions and $f = 0$ for bosons. This follows since, with standard phase conventions, each fermion that is translated from the L th orbital to the 1st one will give rise to a negative sign, as it must be commuted through $(N-1)$ occupied fermion states. This happens exactly p times when the operator T_x is applied q times to the product state associated with $|\tau\rangle$ (see Fig. 1). On the other hand, $T_y^q = (-1)^{pq}$. Again, we evaluate this by acting on $|\tau\rangle$. According to the prescription below Eq. (2), $T_y^q|\tau\rangle = e^{-\frac{2\pi i q}{L} \sum_{j=1}^L n_j} |\tau\rangle = e^{-\frac{2\pi i}{M} \sum_{k=1}^M (u+(k-1)qp)} |\tau\rangle$ where M is the number of unit cells in the pattern such that $L = Mq$, $N = Mp$. The integer $u = \sum_{j=1}^p n_j$ equals the contribution of the first unit cell to the sum such that the k th unit cell contributes $u + (k-1)pq$. Since the u term drops out modulo $2\pi i$, the exponent reads $-i\pi(M-1)pq = i\pi pq \pmod{2\pi i}$, since $Mp = N$ was assumed to be even. Hence $T_y^q = (-1)^{pq}$.

One thus finds that $(-1)^{pq} = (-1)^{fp}$ for any quantum Hall state that can be represented through 1D patterns, whenever all torus ground states are related by translation. This implies that such states satisfy the odd denominator rule: If q were even, p would have to be odd, and the relation would be violated for fermions. It likewise follows that for bosons, out of p and q exactly one needs to be even [19,20]. Within this framework, the odd denominator rule (and its bosonic counterpart) has thus been shown to be a characteristic property of all states with minimum torus degeneracy. It should be noted that for all states in the Abelian hierarchy, bosonic and fermionic, patterns have been worked out in Ref. [21]. It is pleasing to see that with the above, it can be understood from the patterns themselves that some are legitimate for bosonic states only, while some others only qualify for fermionic states.

The strong p -wave pairing state.—Quantum Hall states that satisfy the minimum torus degeneracy are necessarily Abelian. Within the framework of 1D patterns, this follows from the fact that if all patterns are related by translation, domain walls associated with elementary quasiparticle type excitations always generate the same fixed shift between subsequent ground state patterns. In this case, the degeneracy of topological sectors also remains fixed (cf., e.g., [16]), and does not grow exponentially with quasiparticle number as required for non-Abelian states. Conversely, however, Abelian states need not satisfy the minimum torus degeneracy, and can thus violate the odd denominator rule. Examples are found among the Halperin bilayer states, whose thin torus patterns have been given in [6]. While the patterns of such states are not all related by translation, they do all have unit cells of the same size. This must be true in order for the states to be Abelian. A variation in unit cell size between different ground state patterns will always introduce a combinatorial degree of freedom when domain walls between different patterns are formed, as becomes clear, e.g., by considering the Pfaffian case [5,9,13]. In the interpretation of Ref. [22], this always leads to nontrivial fusion rules, implying a non-Abelian state. On the other hand, different patterns of equal unit cell size may or may not do this. With this in mind, it is interesting to ask whether an Abelian single-component state of fermions at $\nu = 1/2$ with an eightfold torus degeneracy can be consistent with the framework described here. These are the quantum numbers relevant to the Abelian state now known as the “strong p -wave pairing” state [23], which was originally discussed in Ref. [24] as a candidate for the plateau at $\nu = 5/2$ [25]. Here we can easily rule out that this state fits into the 1D formalism. The elementary unit cell of the corresponding 1D pattern could not have size 8, for then all eight ground states must be related by translation. This can be ruled out, since the state must then be subject to the odd denominator rule as shown above. Alternatively, an Abelian state at $\nu = 1/2$ could correspond to 1D patterns formed from two different unit cells of size 4, or four different unit cells of size 2. However, this is not possible either, since for fermions at $\nu = 1/2$ there is

only one type of elementary unit cell, modulo translations, of size 4 or size 2. These unit cells are 1100 and 10, respectively. This rules out that any Abelian state with the quantum numbers of the strong pairing state can be described in the language of 1D patterns. In fact, we can also rule out a non-Abelian state with these quantum numbers. Three different ground state patterns of unit cell sizes 4, 2, 2 can be ruled out as in the above. Two ground state patterns of unit cell sizes 6 and 2 are again found to violate the S -duality constraint (4): As shown above, at $\nu = 1/2$ any given pattern, including its translated versions, can only account for 2 different T_y eigenvalues. However, there must be at least 6 different T_x eigenvalues if a pattern has unit cell size 6.

Needless to say, the incompatibility of the strong pairing state with a description in terms of 1D patterns does not rigorously rule out the viability of such a state. It may, however, imply that this state is of a qualitatively different nature compared to other contenders that do allow a 1D labeling, such as the $\nu = 1/2$ Pfaffian. In this regard it is worth noting that so far the strong pairing state seems to have been quite elusive to exact diagonalization studies.

Non-Abelian and other states.—I finally remark that the considerations made above allow one, in simple enough cases, to positively identify the possible quantum Hall states allowed within the 1D formalism, based on the filling factor and the torus degeneracy alone. Indeed, within this framework, these two data may specify the underlying physics quite uniquely. As an example, I will analyze the question of how many possible bosonic quantum Hall states may exist at filling $\nu = 1$ with a threefold torus degeneracy, which fit into the 1D framework. This is easily answered. The pattern 300300... can be ruled out, since it already accounts for a threefold degeneracy. Hence all ground states would be related by translation. However, the state violates the bosonic analogue of the odd denominator rule, and so would then violate Eq. (4). Patterns with unit cell sizes 2 and 1 are unique at $\nu = 1$, and must then constitute the correct ground state patterns. These are 2020... and 1111..., respectively, the patterns associated with the $\nu = 1$ Pfaffian. These satisfy S duality, as already hinted at in Ref. [5]. Furthermore, it has been shown how these patterns essentially encode the statistics of the state [13], modulo a certain Abelian phase. The filling factor $\nu = 1$ and the torus degeneracy 3 thus specify the physics quite uniquely within the framework discussed here. Similar statements can be made about Laughlin states. Moreover, the same constraints also fix the patterns of the level 3 Read-Rezayi state at $\nu = 3/2$ (fourfold degenerate). Here, a single pattern of unit cell size 4 is ruled out: Such a unit cell must contain 6 particles, which is not coprime with 4, in violation of the rules established above. Two patterns of unit cell size 2 is the only other possibility that can account for these quantum numbers. This uniquely determines the patterns to be 3030... and 2121..., which are just the patterns that have been associated to this state

in the literature [10,22]. Last, let us inquire about a state at filling factor $\nu = 2/3$, with torus degeneracy 6. These are the quantum numbers of the bosonic Gaffnian state [26], which, unlike the other states discussed so far, has been proposed to be critical. Irrespective of its physical nature, the associated patterns [10,27] are again unique based on these quantum numbers. At $\nu = 2/3$, possible unit cell sizes must be multiples of 3. A single pattern of unit cell size 6 can be ruled out as in the case of the Read-Rezayi state. There must then be 2 patterns of unit cell size 3. These are again unique, modulo translation: 200... and 101... It remains to be seen if even in this—presumably critical—case, the method of Ref. [13] can be used to ascribe well defined statistics to this state.

I would like to thank A. Karlhede, E. Bergholtz, and D. H. Lee for stimulating discussions. This work was supported by the National Science Foundation under NSF Grant No. DMR-0907793.

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