Fast Lattice Boltzmann Solver for Relativistic Hydrodynamics

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A lattice Boltzmann formulation for relativistic fluids is presented and numerically validated through quantitative comparison with recent hydrodynamic simulations of relativistic fluids. In order to illustrate its capability to handle complex geometries, the scheme is also applied to the case of a three-dimensional relativistic shock wave, generated by a supernova explosion, impacting on a massive interstellar cloud. This formulation opens up the possibility of exporting the proven advantages of lattice Boltzmann methods, namely, computational efficiency and easy handling of complex geometries, to the context of (mildly) relativistic fluid dynamics at large, from quark-gluon plasmas up to supernovae with relativistic outflows.

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We present the first (to the best of our knowledge) lattice Boltzmann (LB) formulation for relativistic fluids. Our procedure is based on two simple and yet apparently unpursued observations: (i) the kinetic formalism is naturally covariant, hyperbolic, or conservative; (ii) being based on construction on a finite-velocity scheme, standard lattice Boltzmann methods [1,2] naturally feature relativisticlike equations of state, whereby the sound speed c_s is a sizable fraction of the speed of light c. For many lattices, $c_s/c = 1/\sqrt{3}$, which is precisely the equation of state of a relativistic ideal gas. Based on the above, we are led to propose that, upon choosing the lattice speed $c_l \equiv \delta x/\delta t \sim c$, the current LB mathematical framework should allow for relativistic extensions, which is indeed the case made in this Letter. This spawns the exciting opportunity of carrying the assets of LB over to the context of mildly relativistic fluids, e.g., the quark-gluon plasma generated by recent experiments on heavy ions and hadron jets [3–9], as well as astrophysical flows, such as interstellar gas and supernova remnants [10,11]. The relativistic lattice Boltzmann (RLB) scheme is validated through quantitative comparison with recent one-dimensional hydrodynamic simulations of relativistic shock-wave propagation in viscous quark-gluon plasmas [12]. However, the same scheme can also be applied to three-dimensional, large-scale relativistic fluid problems, such as the impact of the shock wave generated by, say, a supernova explosion, on a massive cloud, as pictorially shown in Fig. 1.

The present RLB approach is, in principle, limited to weakly relativistic problems, with $|\vec{\beta}| \sim 0.1$. However, by introducing artificial faster-than-light particles (numerical "tachyons"), it is shown to produce quantitatively correct results up to $|\vec{\beta}| \sim 0.6$, corresponding to a Lorentz factor $\gamma = 1/\sqrt{1 - |\vec{\beta}|^2} \sim 1.4$. Although still far from state-of-the-art numerical methods for relativistic hydrodynamics [13,14], the RLB might nevertheless offer a fairly inexpensive alternative to more sophisticated methods at moderate values of $|\vec{\beta}|$. In addition, since LB is recognizedly an excellent solver for flows in complex geometries, like porous media, it is plausible to expect that the present RLB scheme may play a useful role for the simulation of relativistic flows in nonidealized geometries.

We begin by considering the standard relativistic fluid equations associated with the conservation of number of particles and momentum energy, namely, $\partial_\nu T^{\mu\nu} = 0$, where the energy-momentum tensor reads as follows [15,16]: $T^{\mu\nu} = P\eta^{\mu\nu} + (\epsilon + P)u^\mu u^\nu + \pi^{\mu\nu}$, ϵ being the energy density, P the hydrostatic pressure, and $\pi^{\mu\nu}$ the dissipative component of the stress-energy tensor, to be specified later. The velocity 4-vector is defined by

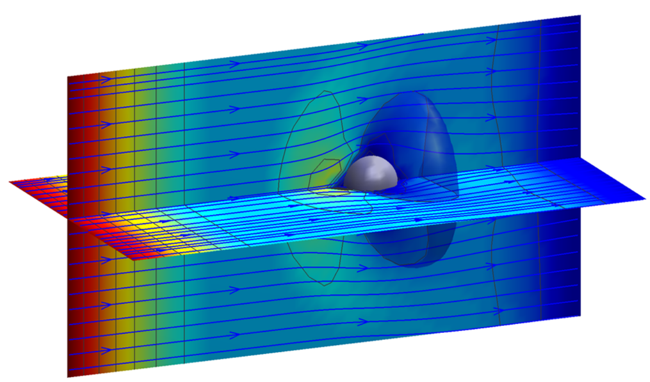

FIG. 1 (color online). Relativistic shock wave, generated by a γ-ray burst or x-ray flash supernova explosion, impacting on a massive interstellar cloud at $|\vec{\beta}| = 0.5$. Here the streamlines represent the velocity field, and the colors the pressure. The simulation was implemented on a grid size of $200 \times 100 \times 100$ cells.

$u^\mu = (\gamma, \gamma\vec{\beta})^\mu$, where $\vec{\beta} = \vec{u}/c$ is the velocity of the fluid in units of the light speed and $\gamma = 1/\sqrt{1 - |\vec{\beta}|^2}$. The tensor $\eta^{\mu\nu}$ denotes the Minkowski metric. Additionally, we define the particle 4-flow $N^\mu = (\gamma n, n\gamma\vec{\beta})^\mu$, with n the number of particles per volume. Applying the conservation rule to energy and momentum, $\partial_\mu T^{\mu\nu} = 0$, and to the 4-flow, $\partial_\mu N^\mu = 0$, we obtain the hydrodynamic equations. Note that, in contrast to a nonrelativistic fluid, we have separate conservation equations for particle number and energy. To complete the set of equations, we need to define a state equation that relates, at least, two of the three quantities: n, P, and ϵ.

The above hydrodynamic equations can be derived as a macroscopic limit of the following relativistic Boltzmann-Bhatnagar-Gross-Krook equation [15,16]: $\partial_\mu(p^\mu f) = \frac{f^{\mathrm{eq}} - f}{c\tau}$, where $p^\mu = [E(p), \vec{p}c]$ is the particle 4-momentum with $E(p)$ the relativistic energy as a function of the momentum modulus $p = |\vec{p}|$, f^{eq} a local relativistic equilibrium, and τ the relaxation time. LB theory for classical fluids shows that it may prove more convenient to solve fluid problems by numerically integrating the underlying kinetic equation rather than the macroscopic fluid equations themselves. The main condition for this to happen is that a sufficiently economic representation of the velocity space degrees of freedom be available. Such a representation is indeed provided by discrete lattices, in which the particle velocity (momentum) is constrained to a handful of constant discrete velocities, with sufficient symmetry to secure the fundamental conservations of fluid flows, namely, mass-momentum-energy conservation as well as rotational invariance.

In order to reproduce the relativistic hydrodynamic equations, we propose a three-dimensional LB model with a 19-speed cell configuration, as shown in Fig. 2. We define two distribution functions f_i and g_i for each velocity vector $\vec{c}_i$, where the index i labels the discrete momenta within each cell. The hydrodynamic variables are calculated by imposing the following macroscopic constraints: $n\gamma = \sum_{i=0}^{18} f_i$, $(\epsilon + P)\gamma^2 - P = \sum_{i=0}^{18} g_i$, and $(\epsilon + P)\gamma^2\vec{u} = \sum_{i=0}^{18} g_i\vec{c}_i$. From these equations, we have to extract six physical quantities, n, $\vec{u}$, ϵ, and P, out of only five equations. The problem is closed by choosing the equation of state for ultrarelativistic fluids, $\epsilon = 3P$.

The distribution functions evolve according to the Bhatnagar-Gross-Krook-Boltzmann evolution equation [17] (full details in a future extended publication),

$$f_i(\vec{x} + \vec{c}_i\delta t, t + \delta t) - f_i(\vec{x}, t) = -\frac{\delta t}{\tau}(f_i - f_i^{\mathrm{eq}}), \quad (1)$$

and

$$g_i(\vec{x} + \vec{c}_i\delta t, t + \delta t) - g_i(\vec{x}, t) = -\frac{\delta t}{\tau}(g_i - g_i^{\mathrm{eq}}), \quad (2)$$

where f_i^{eq} and g_i^{eq} are local equilibrium distribution functions encoding ideal-hydrodynamics information. They read as follows:

$$f_{i\geq 0}^{\mathrm{eq}} = w_i n\gamma\left[1 + 3\frac{(\vec{c}_i\cdot\vec{u})}{c_l^2}\right], \quad (3)$$

$$g_{i>0}^{\mathrm{eq}} = 3w_i\left[\frac{(\epsilon + P)\gamma^2}{c_l^2}\left[(\vec{c}_i\cdot\vec{u}) + \frac{3(\vec{c}_i\cdot\vec{u})^2}{2c_l^2} - \frac{|\vec{u}|^2}{2}\right] + \frac{P}{c_l^2}\right], \quad (4)$$

$$g_{i=0}^{\mathrm{eq}} = 3w_0(\epsilon + P)\gamma^2\left[1 - \frac{P(2 + c_l^2)}{(P + \epsilon)\gamma^2 c_l^2} - \frac{1}{2}\frac{|\vec{u}|^2}{c_l^2}\right]. \quad (5)$$

Here c_l is the limiting velocity of the lattice, which relates the cell size and the time step $c_l = \frac{\delta x}{\delta t}$, and we have rescaled the velocity units such that the speed of light $c = 1$. The weights for this set of discrete speeds are defined by $w_0 = 1/3$ for the rest particles, $w_i = 1/18$ for the velocities $|\vec{c}_i| = c_l$, and $w_i = 1/36$ for $|\vec{c}_i| = \sqrt{2}c_l$.

By Taylor expanding the right-hand side of (1) and (2) to second order in δt, and retaining terms only up to first order in the Chapman-Enskog expansion $f = f^{\mathrm{eq}} + kf^1 + \cdots$, where $k \sim c\tau\nabla$ is the Knudsen number, the LB equations can be shown to reproduce the following continuum fluid equations:

$$\partial_t[(\epsilon + P)\gamma^2 - P] + \partial_i[(\epsilon + P)\gamma^2 u_i] = 0, \quad (6a)$$

$$\partial_t[(\epsilon + P)\gamma^2 u_i] + \partial_i P + \partial_j[(\epsilon + P)\gamma^2 u_i u_j] = \partial_j[\partial_i(\eta\gamma u_j) + \partial_j(\eta\gamma u_i) + \partial_k(\eta\gamma u_k)\delta_{ij}], \quad (6b)$$

for the energy-momentum conservation, and

$$\partial_t(n\gamma) + \partial_i(n\gamma u_i) = 0, \quad (7)$$

for the conservation of particle number. The indices i, j, and k denote the spatial components. The shear viscosity, computed according to standard LB procedures, is $\eta = \frac{1}{3}\gamma(\epsilon + P)(\tau - \delta t/2)c_l^2$. Note that the LB equations are inherently dissipative, since linear stability imposes the condition $\delta t < 2\tau$, i.e., $\eta > 0$. Most remarkably, the negative shift $-\delta t/2$, which stems directly from the light-cone structure of the LB streaming operator, permits one to attain very small viscosities, of order, say, 10^{-3} in lattice units, while still keeping $\delta t = 1$, and $\tau \sim 1/2 + O(10^{-3})$. This permits the simulation of very-low viscous flows (such as the quark-gluon plasma) with time steps of order $O(1)$, which proves very beneficial for computational purposes. Another valuable property of the LB formulation is that, in contrast to other hydrodynamic formulations, dissipation is not represented explicitly through second-order

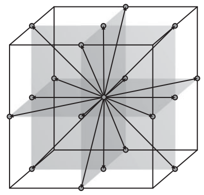

FIG. 2. Set of discrete velocities for the relativistic lattice Boltzmann model. The highest speed is $\sqrt{2}c_l$.

spatial derivatives but emerges instead from first-order, covariant propagation-relaxation dynamics, through adiabatic enslaving of the momentum-flux tensor to its equilibrium (ideal-hydrodynamic) expression. As a result of this first-order dynamics, the Courant-Friedrichs-Lewy stability condition of the LB scheme reads simply as $u\delta t \leq \delta x$, instead of $\eta \delta t < \delta x^2$, the latter being much more demanding on the time step δt, as the grid is refined ($\delta x \to 0$). Also, it is worth noting that our scheme smoothly recovers the nonrelativistic limit by simply letting $\beta \to 0$.

To test the model we solve the Riemann problem in viscous gluon matter [12]. We use the relation between energy density and particle number density, $\epsilon = 3nT$, T being the flow temperature [15]. The initial configuration consists of two regions divided by a membrane located at $z = 0$. Both regions have thermodynamically equilibrated matter with different constant pressure P_0 for $z < 0$ and P_1 for $z > 0$. At $t = 0$ the membrane is removed. We implement a one-dimensional simulation with an array of size $1 \times 1 \times 800$. In this case, the 4-velocity is given by $u^\mu = (\gamma, 0, 0, \gamma\beta)^\mu$. From Eqs. (3) and (4), it is seen that the positivity condition $f_i^{\text{eq}} > 0$ implies $\vec{c}_i \cdot \vec{u} < \frac{c_l^2}{3}$. Thus, by increasing c_l, the LB scheme is protected against positivity-violating numerical instabilities. Clearly, such an expedient must be accompanied by a corresponding reduction of the time step δt, to preserve the light-cone condition $c_l \delta t = \delta x$. For mildly relativistic problems, say, $\beta \sim 0.2$, we choose $c_l = c$, i.e., $c = 1$ in lattice units, so that, strictly speaking, particles propagating along diagonal links (see Fig. 2), move faster than light by a factor $\sqrt{2}$ (numerical tachyons). Since the physically relevant signal is not the particle speed $\vec{c}_i$, but the fluid one $\vec{u}$, at least for mildly relativistic problems, this is a perfectly viable procedure.

Based on the above, we choose $\delta x = 0.008$ fm and $\delta t = 0.008$ fm$/c$ in physical units. The entropy density is computed as $s = 4n - n \ln\lambda$, with $\lambda = \frac{n}{n^{\text{eq}}}$ the gluon fugacity and the equilibrium particle density n^{eq} given by $n^{\text{eq}} = \frac{d_G T^3}{\pi^2}$, with $d_G = 16$ for gluons. Next, we can calculate the ratio between the viscosity and entropy density, η/s, that is used as a parameter to characterize the emergence of shock waves within the quark-gluon plasma. Pressures were chosen as $P_0 = 5.43$ GeV fm^{-3} and $P_1 = 2.22$ GeV fm^{-3}, corresponding to 7.9433×10^{-6} and 3.2567×10^{-6} lattice units, respectively. The initial temperature is $T_0 = 350$ MeV, corresponding to $T_0 = 0.0287$ lattice units. With these parameters, the conversion between physical and numerical units for the energy is 1 MeV $= 8.2 \times 10^{-5}$. Figure 3 shows the results for different values of η/s and the comparison with the BAMPS (Boltzmann approach of multiparton scattering) [18] microscopic transport model simulations [12] at time 3.2 fm$/c$. On the other hand, in Fig. 4, we can see the evolution of the system for $\eta/s = 0.1$ by comparing the two numerical models. In both cases, we find an excellent agreement with BAMPS. To simulate fluids moving at $\beta \sim 0.6$, we use $c_l = 10$. The pressure P_1 is taken as 0.9532 GeV fm^{-3}, and we define two temperatures $T_0 = 0.0328$ and $T_1 = 0.0164$, the first one for $z < 0$ and the second one for $z > 0$. Figure 5 shows the shock wave for $\eta/s = 0.001$, and the comparison with the BAMPS simulation [12] is again excellent.

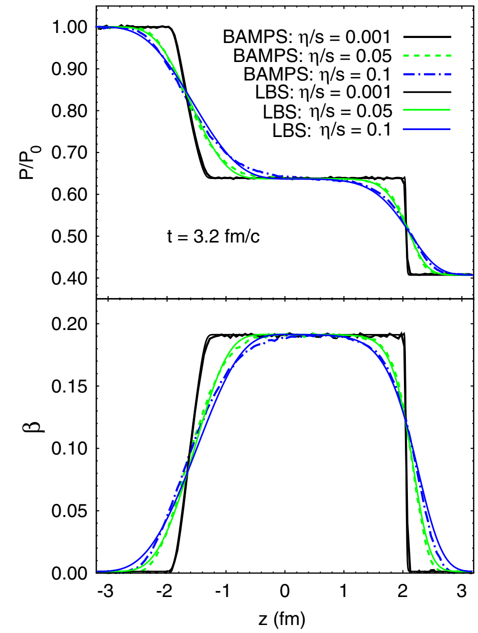

FIG. 3 (color online). Comparison between the BAMPS simulations [12] and the LB results, for $\beta \sim 0.2$. Pressure (top) and velocity (bottom) of the fluid as a function of the spatial coordinate z.

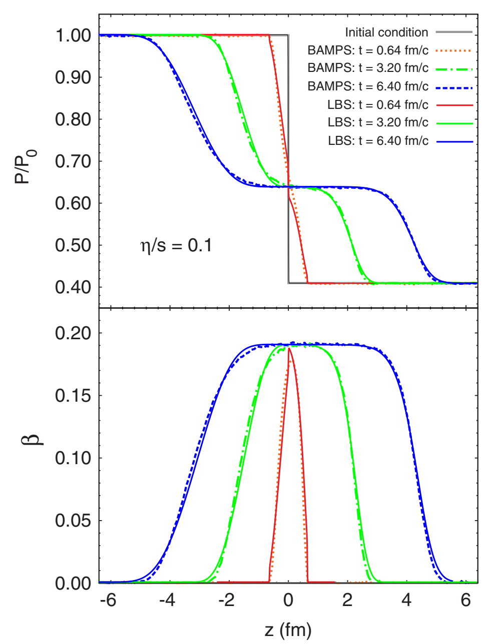

FIG. 4 (color online). Time evolution of the shock wave for BAMPS simulations [12] and LB results, $\beta \sim 0.2$.

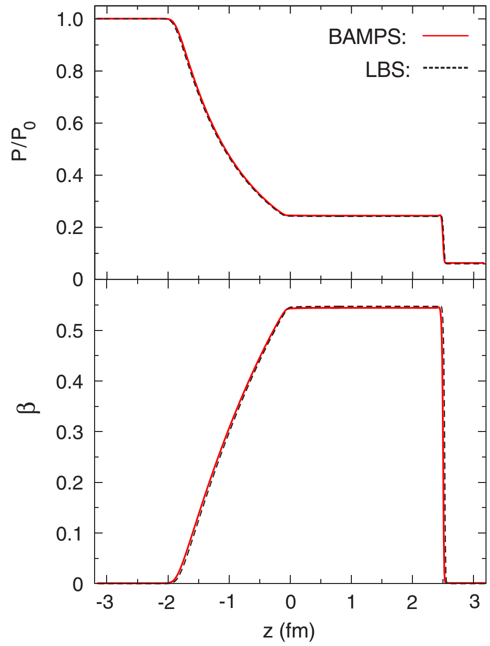

FIG. 5 (color online). Velocity and pressure profile using numerical tachyons, $c_l = 10$, at time $t = 3.2$ fm$/c$, with $\beta \sim 0.6$ and $\eta/s = 0.001$.

Our LB scheme easily extends to three dimensions, as illustrated in Fig. 1, where we simulate a shock wave, generated by, say, a γ-ray burst or x-ray flash supernova explosion, colliding against a massive interstellar cloud. The ejecta from the explosion of such supernovae are known to sweep the interstellar material up to relativistic velocities along the way (relativistic outflows) [10]. A typical $200 \times 100 \times 100$ lattice-site simulation spanning 1350 time steps takes about 1900 CPU seconds on a standard PC. Although a one-to-one comparison remains to be done, our hydrokinetic algorithm appears to be nearly an order of magnitude faster than corresponding hydrodynamic codes. This is most likely due to the fact that, in the LB representation, information travels on constant light cones rather than on material fluid streamlines [19]. As a result, the Riemann problem trivializes to a mere shift of the distribution function along the corresponding light cone, a floating-point free, exact operation. Moreover, the parametric scans conducted so far have not exposed any numerical instability problem. A more exhaustive study along these lines will be presented in a future and lengthier publication.

In summary, we have developed a LB formulation for (mildly) relativistic fluids, with β up to $\sim$0.6. The scheme exhibits excellent agreement with previous numerical simulations of shock-wave propagation in quark-gluon plasmas, evidently at a fraction of the cost of hydrodynamic codes. The present RLB scheme shows promise to offer an efficient numerical solver for complex shock-propagation and collision events of direct interest to relativistic fluid dynamics at large, from quark-gluon plasmas to large-scale astrophysical flows in nonidealized geometries. In analogy to compressible (nonrelativistic) fluids, resorting to higher-order polynomial equilibria, and correspondingly higher-symmetry lattices, is expected to soften the positivity constraints, thereby giving access to still higher values of β. These and other developments, such as entropic formulations [20] and the extension to nonideal equations of state, will be the object of future work.

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