Precision Quantum Metrology and Nonclassicality in Linear and Nonlinear Detection Schemes

Ángel Rivas[1] and Alfredo Luis[2,*]

[1]Institut für Theoretische Physik, Universität Ulm, Ulm D-89069, Germany

[2]Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain

(Received 14 February 2010; published 30 June 2010)

We examine whether metrological resolution beyond coherent states is a nonclassical effect. We show that this is true for linear detection schemes but false for nonlinear schemes, and propose a very simple experimental setup to test it. We find a nonclassicality criterion derived from quantum Fisher information.

DOI: 10.1103/PhysRevLett.105.010403 PACS numbers: 03.65.Ca, 03.65.Ta, 42.50.Ar, 42.50.Dv

Nonclassicality is a key concept supporting the necessity of the quantum theory. There is widespread consensus that the coherent states $|\alpha\rangle$ are the classical side of the borderline between the quantum and classical realms [1,2]. In quantum metrology it is usually believed that resolution beyond coherent states is a quantum effect, since this is achieved by famous nonclassical probe states, such as squeezed, number, or coherent superpositions of distinguishable states [3]. However, this does not mean that every state providing larger resolution than coherent states is nonclassical.

In this Letter we test this belief by examining whether metrological resolution beyond coherent states is necessarily a nonclassical effect or not [4]. To this end we find a novel nonclassicality criterion derived from quantum Fisher information. We demonstrate that the belief is true for linear detection schemes but false for nonlinear schemes. Nonlinear detection is a recently introduced item in quantum metrology that has plenty of promising possibilities and is being thoroughly studied and implemented in different areas such as quantum optics [5,6], Bose-Einstein condensates [7,8], nanomechanical resonators [9], and atomic magnetometry [10].

Throughout we focus on single-mode quantum light beams with complex amplitude operators a such that $[a, a^\dagger] = 1$ and $a|\alpha\rangle = \alpha|\alpha\rangle$. Resolution provided by different probe states is compared for the same mean number of photons $\bar{n}$ that represents the energy resources available for the measurement. We examine the following proposition.

Proposition.—A probe state ρ providing larger resolution than coherent states $|\alpha_\rho\rangle$ with the same mean number of photons $\bar{n}$ is nonclassical, where

$$\bar{n} = \langle\alpha_\rho|a^\dagger a|\alpha_\rho\rangle = |\alpha_\rho|^2 = \mathrm{tr}(\rho a^\dagger a). \tag{1}$$

A customary signature of nonclassical behavior is the failure of the Glauber-Sudarshan $P(\alpha)$ phase-space representation to exhibit all the properties of a classical probability density [1]. This occurs when $P(\alpha)$ takes negative values, or when it becomes more singular than a delta function. To test the proposition we must specify how resolution is assessed.

Resolution.—In a detection scheme the signal to be detected χ is encoded in the input probe state ρ by a transformation $\rho \rightarrow \rho_\chi$. For definiteness, we focus on the most common and practical case of unitary transformations with constant generator G independent of the parameter

$$\rho_\chi = \exp(i\chi G)\rho\exp(-i\chi G). \tag{2}$$

The value of χ is inferred from the outcomes of measurements performed on ρ_χ. The ultimate resolution of such inference is given by the quantum Fisher information $F_Q(\rho_\chi)$ since the variance of any unbiased estimator $\tilde{\chi}$ is bounded from below in the form [11,12]

$$(\Delta\tilde{\chi})^2 \geq \frac{1}{NF_Q(\rho_\chi)}, \tag{3}$$

where N is the number of independent repetitions of the measurement.

Better resolution is equivalent to larger quantum Fisher information, which can be expressed as [12,13]

$$F_Q(\rho_\chi) = 2\sum_{j,k}\frac{(r_j - r_k)^2}{r_j + r_k}|\langle r_j|G|r_k\rangle|^2, \tag{4}$$

where $|r_j\rangle$ are the eigenvectors of ρ with eigenvalues r_j and the sum includes all the cases with $r_j + r_k \neq 0$. So for uniparametric unitary transformations F_Q is independent of χ [13].

In order to reach ultimate sensitivity predicted by the quantum Fisher information, an optimum measurement and an efficient estimator are required [12]. If we consider the maximum likelihood as estimator, the number of repetitions required to reach the efficient regime may depend on the probe state [14]. In order to focus on the intrinsic capabilities of different schemes, we will assume that N is large enough so that optimum conditions are reached for all cases, so that schemes are compared by comparing their quantum Fisher information. Note also that resolution depends also on the duration of the measurement. Because of this any meaningful comparison between different schemes should be done on equal-time basis.

Let us show three useful properties of the quantum Fisher information. (i) For pure states, such as coherent states $|\alpha\rangle$, the quantum Fisher information becomes proportional to the variance of the generator [12]

$$F_Q(|\alpha\rangle, G) = 4(\Delta_\alpha G)^2 = 4(\langle\alpha|G^2|\alpha\rangle - \langle\alpha|G|\alpha\rangle^2). \quad (5)$$

(ii) The quantum Fisher information is convex. For a proof based on the monotonocity of quantum Fisher information under complete positive maps, see Ref. [15]. A much simpler proof is given by a straightforward use of the convexity of the Fisher information and the Braunstein-Caves inequality [12]. Thus, for classical states

$$\rho_{\text{class}} = \int d^2\alpha P_{\text{class}}(\alpha)|\alpha\rangle\langle\alpha|, \quad (6)$$

where $P_{\text{class}}(\alpha)$ is a non-negative function no more singular than a delta function, convexity implies the following bound for the quantum Fisher information of classical states:

$$F_Q(\rho_{\text{class}}, G) \leq \int d^2\alpha P_{\text{class}}(\alpha)F_Q(|\alpha\rangle, G) = 4\int d^2\alpha P_{\text{class}}(\alpha)(\Delta_\alpha G)^2. \quad (7)$$

(iii) In most cases it is rather difficult to compute analytically $F_Q(\rho, G)$, especially in infinite dimensional systems. A similar but simpler performance measure is

$$\Lambda^2(\rho, G) = \text{tr}(\rho^2 G^2) - \text{tr}(\rho G \rho G) \quad (8)$$

or, equivalently, in the same conditions of Eq. (4),

$$\Lambda^2(\rho, G) = \frac{1}{2}\sum_{j,k}(r_j - r_k)^2|\langle r_k|G|r_j\rangle|^2, \quad (9)$$

which for pure states such as coherent states also becomes the variance of the generator $\Lambda(|\alpha\rangle, G) = \Delta_\alpha G$ [16]. This is derived from the Hilbert-Schmidt distance between ρ_χ and ρ in the same terms in which the quantum Fisher information is derived from the Bures distance [12,17]. The useful point here is that from Eqs. (4) and (9) and given that $r_k + r_l \leq 1$ it holds that

$$F_Q(\rho, G) \geq 4\Lambda^2(\rho, G), \quad (10)$$

the equality being reached for pure states.

Nonclassicality from quantum Fisher information.—For the sake of convenience let us express the variance of G on coherent states as a mean value

$$(\Delta_\alpha G)^2 = \langle\alpha|A_G|\alpha\rangle, \qquad A_G = G^2 - {:}G^2{:}, \quad (11)$$

where :: denotes normal order, and G in ${:}G^2{:}$ must be expressed in its normally ordered form so that $\langle\alpha|{:}G^2{:}|\alpha\rangle = \langle\alpha|G|\alpha\rangle^2$. A key point is that $\langle\alpha|A_G|\alpha\rangle$ gives the quantum Fisher information of coherent states,

$$F_Q(|\alpha\rangle, G) = 4\langle\alpha|A_G|\alpha\rangle, \quad (12)$$

so that the bound (7) for the quantum Fisher information of classical states reads

$$F_Q(\rho_{\text{class}}, G) \leq 4\int d^2\alpha P_{\text{class}}(\alpha)\langle\alpha|A_G|\alpha\rangle = 4\,\text{tr}(\rho_{\text{class}}A_G). \quad (13)$$

This relation is derived from the convexity of $F_Q(\rho, G)$, so it relies entirely on the classical nature of $P_{\text{class}}(\alpha)$. Therefore its violation provides the following nonclassicality criterion:

$$F_Q(\rho, G) > 4\,\text{tr}(\rho A_G) \rightarrow \rho \text{ is nonclassical.} \quad (14)$$

Since this criterion is formulated in terms of the quantum Fisher information, it will be useful to discuss the interplay between improved metrological resolution and nonclassicality. The key point is to link $\text{tr}(\rho A_G)$ in the nonclassical criterion (14) with the quantum Fisher information of coherent states with the same mean number of photons $F_Q(|\alpha_\rho\rangle, G) = 4\langle\alpha_\rho|A_G|\alpha_\rho\rangle$. This is straightforward when $A_G \propto a^\dagger a$. To study this in detail let us split the analysis in linear and nonlinear schemes.

Linear schemes.—By linear schemes we mean that the signal is encoded via input-output transformations where the output complex amplitudes are linear functions of the input ones and their conjugates. Their generators are polynomials of a, $a^\dagger$ up to second order, embracing all traditional interferometric techniques exemplified by the phase shifts generated by the photon-number operator

$$G = A_G = a^\dagger a, \quad (15)$$

so that G and A_G coincide. In this case the resolution (quantum Fisher information) provided by coherent probe states is given by its mean number of photons

$$F_Q(|\alpha_\rho\rangle, a^\dagger a) = 4\langle\alpha_\rho|a^\dagger a|\alpha_\rho\rangle = 4|\alpha_\rho|^2 = 4\,\text{tr}(\rho a^\dagger a), \quad (16)$$

where we have used Eqs. (1), (12), and (15). The probe states ρ providing larger resolution than coherent states $|\alpha_\rho\rangle$ with the same mean number of photons satisfy

$$F_Q(\rho, a^\dagger a) > F_Q(|\alpha_\rho\rangle, a^\dagger a) = 4\,\text{tr}(\rho a^\dagger a), \quad (17)$$

so that from the nonclassical criterion (14) they are necessarily nonclassical states and the proposition being tested is true.

This result also holds for other generators of linear transformations such as $G = a\exp(i\theta) + a^\dagger\exp(-i\theta)$, which generates displacements of the quadratures, and $G = a^2\exp(i\theta) + a^{\dagger 2}\exp(-i\theta)$, which generates quadrature squeezing, where θ is an arbitrary phase [18]. This is because $A_G = 1$ and $A_G = 4a^\dagger a + 2$, respectively, so that $4\,\text{tr}(\rho A_G) = F_Q(|\alpha_\rho\rangle, G)$.

This also holds for two-mode SU(2) generators

$$G = \boldsymbol{u}\cdot\boldsymbol{J}, \qquad A_G = a_1^\dagger a_1 + a_2^\dagger a_2, \quad (18)$$

where $\boldsymbol{u}$ is a three-dimensional unit real vector and $\boldsymbol{J}$ are the bosonic realization of the angular momentum operators that generate the SU(2) group

$$J_x = a_1^\dagger a_2 + a_1 a_2^\dagger, \qquad J_y = i(a_1^\dagger a_2 - a_1 a_2^\dagger), \qquad J_z = a_1^\dagger a_1 - a_2^\dagger a_2. \tag{19}$$

This describes all two-beam lossless optical devices, such as beam splitters, phase plates, and two-beam interferometers. In this two-mode context the coherent states $|\alpha\rangle$ refer to the product of single-mode coherent states $|\alpha\rangle = |\alpha_1\rangle|\alpha_2\rangle$ with mean number of photons $\bar{n} = |\alpha_1|^2 + |\alpha_2|^2 = \langle\alpha|A_G|\alpha\rangle$. For a simple derivation of A_G in Eq. (18), note that any $\boldsymbol{u}\cdot\boldsymbol{J}$ is in normal order, normal order commutes with SU(2) transformations, $\boldsymbol{u}\cdot\boldsymbol{J}$ is SU(2) equivalent to J_z, with $J_z^2 - :J_z^2: = a_1^\dagger a_1 + a_2^\dagger a_2$, and $a_1^\dagger a_1 + a_2^\dagger a_2$ is SU(2) invariant. This is $A_{UGU^\dagger} = UA_G U^\dagger$ if G is in normal order and U is a SU(2) unitary transformation.

When the angular momentum $\boldsymbol{J}$ refers collectively to a system of qubits, it has been demonstrated [19] that improved resolution beyond coherent states implies entanglement between qubits. We recover this result by noticing that spin nonclassicality is equivalent to entanglement [20]. This equivalence no longer holds when entanglement refers to the entanglement between field modes; this is to say that nonclassical factorized states $|\psi_1\rangle|\psi_2\rangle$, where $|\psi_j\rangle$ is in mode a_j, can provide better resolution than coherent states.

Nonlinear schemes.—By nonlinear detection schemes we mean that the signal is encoded via input-output transformations where the output complex amplitudes are not linear functions of the input ones. A suitable example is given by

$$G = (a^\dagger a)^2, \qquad A_G = 4a^{\dagger 3}a^3 + 6a^{\dagger 2}a^2 + a^\dagger a, \tag{20}$$

and the key point is that A_G is no longer proportional to the number operator. In practical quantum-optical terms this corresponds to light propagation through nonlinear Kerr media [1].

Next we show that there are classical states that provide larger resolution than coherent states with the same mean number of photons, so that the proposition being tested is false. To this end let us consider the mixed probe state

$$\rho_{\text{class}} = p|\alpha/\sqrt{p}\rangle\langle\alpha/\sqrt{p}| + (1-p)|0\rangle\langle 0|, \tag{21}$$

where $|\alpha/\sqrt{p}\rangle$ is a coherent state, $|0\rangle$ is the vacuum, and $1 > p > 0$. The state ρ_{class} has the same mean number of photons as the coherent state $|\alpha\rangle$ for every p.

Since in general $F_Q(\rho_{\text{class}}, G)$ is difficult to compute when ρ_{class} is mixed, we resort to Eq. (10) so that if

$$4\Lambda^2(\rho_{\text{class}}, G) > F_Q(|\alpha\rangle, G), \tag{22}$$

then $F_Q(\rho_{\text{class}}, G) > F_Q(|\alpha\rangle, G)$ and ρ_{class} provides larger resolution than $|\alpha\rangle$. Using Eq. (8) the condition (22) is equivalent to the following relation between variances of G in coherent states

$$p^2(\Delta_{\alpha/\sqrt{p}}G)^2 > (\Delta_\alpha G)^2, \tag{23}$$

where we have used $|0\rangle$ as an eigenstate of G with null eigenvalue. After Eqs. (11) and (20)

$$(\Delta_\alpha G)^2 = 4|\alpha|^6 + 6|\alpha|^4 + |\alpha|^2, \tag{24}$$

and from Eq. (23) the state ρ_{class} provides larger resolution than $|\alpha\rangle$ provided that $|\alpha|^2 > \sqrt{p}/2$, which can be easily fulfilled.

We are able to observe this improvement even with a very simple and practical measuring scheme such as homodyne detection illustrated in Fig. 1. For that we evaluate the Fisher information $F_C(\rho_{\text{class}}, G)$ of the measurement for ρ_{class} in Eq. (21),

$$F_C(\rho_{\text{class}}, G) = \int dx \frac{1}{P(x|\chi)}\left(\frac{\partial P(x|\chi)}{\partial\chi}\right)^2, \tag{25}$$

where $P(x|\chi) = \langle x|\rho_\chi|x\rangle$ is the probability of the outcome x of the X quadrature, with $X = a^\dagger + a$ and $X|x\rangle = x|x\rangle$. We consider very small χ so that the classical Fisher information is evaluated at $\chi = 0$. We also assume an optimum value for the phase of the coherent amplitude $\alpha = i\sqrt{\bar{n}}$. Using the results in Ref. [6] we get for large $\bar{n}$

$$F_C(\rho_{\text{class}}, G) = 16\frac{\bar{n}^3}{p} = \frac{1}{p}F_C(|\alpha\rangle, G). \tag{26}$$

Thus, the Fisher information for the classical probe state ρ_{class} is above the value for the coherent states with the same mean number of photons $|\alpha\rangle$, especially when $p \to 0$.

Discussion.—To some extent this may be regarded as a paradoxical result, especially in the limit $p \to 0$ where ρ_{class} tends to be the vacuum, $\langle 0|\rho_{\text{class}}|0\rangle \to 1$, since the vacuum state is useless for detection. Nevertheless next we show that this is a fully meaningful and worthy result. To this end let us consider that we repeat the measurement N times with the probe ρ_{class} in Eq. (21). That will be equivalent to get Np times the result of the probe state $|\alpha/\sqrt{p}\rangle$ and $N(1-p)$ times the useless vacuum. Therefore the useful resources are $N|\alpha|^2$ photons distributed in Np runs of $|\alpha|^2/p$ photons. When the probe is $|\alpha\rangle$ (this is the case $p = 1$), all runs are useful and we get the same resources $N|\alpha|^2$ distributed in N runs of $|\alpha|^2$ photons. For linear detection schemes the two allocations of resources

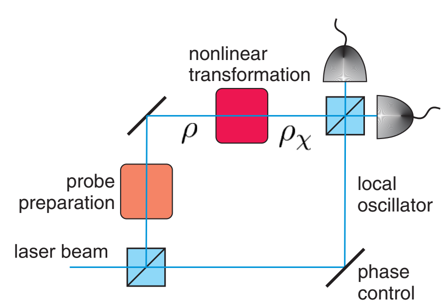

FIG. 1 (color online). Sketch of a homodyne measurement.

provide essentially the same resolution for every p because for large number of photons $\langle\alpha|0\rangle \simeq 0$ it holds that $F_Q(\rho_{\text{class}}, a^\dagger a) \simeq pF_Q(|\alpha/\sqrt{p}\rangle, a^\dagger a) = F_Q(|\alpha\rangle, a^\dagger a)$. However, the nonlinearity greatly privileges large photon numbers so that the best strategy is to put as many photons as possible in a single run, instead of splitting them into several runs. More specifically, for large $|\alpha|$ it holds that $\langle\alpha|0\rangle \simeq 0$ and $F_Q(\rho_{\text{class}}, G) \simeq 16|\alpha|^6/p^2$ while $F_Q(|\alpha\rangle, G) \simeq 16|\alpha|^6$ so that ρ_{class} provides much larger resolution than $|\alpha\rangle$ as $p \to 0$.

Incidentally, the above calculus shows that when $\langle\alpha|0\rangle \simeq 0$ we get $F_C(\rho_{\text{class}}, G) \simeq pF_Q(\rho_{\text{class}}, G)$. This is to say that whereas both $F_{C,Q}$ increase when p decreases, it holds that F_Q increases faster than F_C.

Finally, it might be argued that the improvement of resolution in nonlinear schemes, and the differences between different classical input probes just discussed, may be ascribed to nonclassicality induced by nonlinear transformations. We can rule out this possibility. The quantum Fisher information does not depend on the value of the signal, so that the optimum sensitivity cannot depend on the amount of nonclassicality induced by the transformation. In particular, for the usual case of small signals the induced nonclassicalities will be negligible.

Conclusions.—We have obtained a general nonclassical test derived from quantum Fisher information. For linear detection schemes this test demonstrates that improved resolution beyond coherent states is a nonclassical feature. For nonlinear schemes the situation is different since mixed classical states can provide better resolution than coherent states. This result is very attractive since the key point of classical states is that they are extremely robust against experimental imperfections [6,8] and they are easy to generate in labs.

A. R. acknowledges Susana F. Huelga for illuminating comments and financial support from the EU Integrated Projects QAP, QESSENCE, and the STREP action CORNER. A. L. acknowledges support from official Spanish projects No. FIS2008-01267 and QUITEMAD S2009-ESP-1594.

*alluis@fis.ucm.es

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