Stokes Phenomenon and Schwinger Vacuum Pair Production in Time-Dependent Laser Pulses

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Particle production due to external fields (electric, chromoelectric, or gravitational) requires evolving an initial state through an interaction with a time-dependent background, with the rate being computed from a Bogoliubov transformation between the in and out vacua. When the background fields have temporal profiles with substructure, a semiclassical analysis of this problem confronts the full subtlety of the Stokes phenomenon: WKB solutions are only local, while the production rate requires global information. We give a simple quantitative explanation of the recently computed [Phys. Rev. Lett. **102**, 150404 (2009)] oscillatory momentum spectrum of e^+e^- pairs produced from vacuum subjected to a time-dependent electric field with subcycle laser pulse structure. This approach also explains naturally why for spinor and scalar QED these oscillations are out of phase.

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The Schwinger effect, the nonperturbative production of electron-positron pairs from vacuum in an external electric field, is a highly nontrivial prediction of QED [1–3], but the physical scales are such that it is so weak that it has not yet been directly observed. Recent experimental advances [4] have raised hopes that lasers may achieve fields just 1 or 2 orders of magnitude below the critical field strength of $E_{\rm cr} \sim 10^{16}$ V/cm, either in optical high-intensity laser facilities such as HiPER (Rutherford Laboratory) and the Extreme Light Infrastructure (ELI), or in x-ray free electron laser facilities. Theoretically, recent analyses suggest that the nonperturbative Schwinger effect may be observable at these lower field strengths, by careful shaping and combining of laser pulses leading to a "dynamically assisted Schwinger mechanism" enhancement [5–10]. The most important message is that the detailed shape of the laser pulse is significant, which motivates the extension presented here of the standard WKB approach to include more realistic laser field profiles.

Observation of the Schwinger effect in the nonperturbative domain has the potential to yield valuable insight into analogous gravitational effects [11,12], such as Unruh and Hawking radiation, where direct experiments are not feasible, and where issues such as backreaction and out-of-equilibrium physics are poorly understood. The basic physics is also relevant for atomic, molecular, astrophysics, and plasma physics with ultrahigh intensity lasers, where nonperturbative effects are crucial [13,14], for heavy ion collisions [15], and for the Landau-Zener effect.

We model the electric field in the focal region of two counterpropagating laser pulses by a spatially homogeneous electric field $\vec{E}(t) = (0, 0, E(t))$, with vector potential $\vec{A}(t) = (0, 0, A(t))$, such that $E(t) = -\dot{A}(t)$ with

$$E(t) = E_0 \cos(\omega t + \phi) \exp[-t^2/(2\tau^2)]. \tag{1}$$

Even in this approximation where we neglect spatial focusing, the laser field may involve many physical scales, leading to interesting new phenomena [16]. Here ω is the laser frequency, τ defines the pulse length, and ϕ is the "carrier-phase" offset. The first surprising result in [16] was that the longitudinal momentum spectrum of the produced electron-positron pairs is extremely sensitive to the value of $\omega\tau$, even when $\phi = 0$. For $\omega\tau \gtrsim 4$, the momentum spectrum exhibits oscillations, and these become dramatically enhanced as ϕ increases, to the point where at $\phi = \pi/2$ the spectrum develops minima with zero produced pairs (see Fig. 4 in [16]). The second surprising result in [16] was that the oscillatory minima and maxima are interchanged between spinor and scalar QED. By contrast, when computing the *total* pair production rate (obtained by an integral over the momenta), one conventionally approximates the case of real spinor QED by scalar QED, with an overall multiplicative spin factor of 2. A direct application of the usual semiclassical "imaginary time method" (ITM) [17–21] to this problem does not account for these oscillations, let alone for the difference of phase between spinor and scalar QED. Here we show that the Stokes phenomenon gives a *quantitative* semiclassical explanation of both these effects.

The essential physical interpretation of these oscillations is a resonance effect in the corresponding quantum mechanical scattering problem [16]. This same physical explanation has also been noted for the photoelectron spectrum in atomic ionization [19], where such oscillations have been observed [22,23]. We turn this physical picture into a quantitative method. This should also be relevant for the matterless double-slit experiment [16,24]. Recall that with a time-dependent electric field, the pair production process can be reduced to a one-dimensional over-the-barrier "quantum mechanical" scattering problem [3,17,18], with effective "Schrödinger equation" (in t rather than x)

$$\ddot{\phi} + Q^2(t)\phi = 0, \qquad Q^2(t) \equiv m^2 + p_\perp^2 + [p - A(t)]^2 \tag{2}$$

coming from the Klein-Gordon equation for the particle in the presence of the laser field. (We first discuss scalar QED and later come to spinor QED, where the relevant equation is the Dirac equation.) It is an over-the-barrier scattering problem since the "potential", $-[p - A(t)]^2$, is negative, while the "energy" $(m^2 + p_\perp^2)$ is positive. Thus the reflection coefficient, from which we deduce the probability of pair production, is exponentially small in the semiclassical regime where $E_0 \ll m^2$. This scattering problem can be solved using WKB methods, or numerically using the quantum kinetic approach [16,25] or direct integration of the scattering problem [18]. The equivalence between these approaches is explained in [26].

The Bogoliubov transformation approach [17–21,25] is based on the field decomposition

$$\phi = \frac{\alpha}{\sqrt{2Q}} e^{-i\int^t Q} + \frac{\beta}{\sqrt{2Q}} e^{i\int^t Q},$$
$$\dot{\phi} = -iQ\left(\frac{\alpha}{\sqrt{2Q}} e^{-i\int^t Q} - \frac{\beta}{\sqrt{2Q}} e^{i\int^t Q}\right), \qquad (3)$$

which enforces equations relating the coefficient functions α and β. Unitarity requires $|\alpha|^2 - |\beta|^2 = 1$, and the particle number momentum spectrum is $N(\vec{p}) = |\beta_{\vec{p}}(t = \infty)|^2$, related to the reflection coefficient $|R|^2 = |\beta|^2/(1 + |\beta|^2)$. The Stokes phenomenon is relevant because in calculating $|R|^2$ we compare $\beta(t = \infty)$ to $\beta(t = -\infty)$. However, the leading WKB solutions, $e^{\pm i\int Q}/\sqrt{2Q}$, on which (3) is based, are multivalued functions, only defined *locally*. Evolving a semiclassical approximation from $t = -\infty$ to $t = +\infty$, we cross Stokes and anti-Stokes lines, lines along which $e^{\pm i\int^t Q}$ are exponential or oscillatory, respectively. On crossing such lines, we must take care to keep track properly of the dominant and subdominant solutions. This is the Stokes phenomenon [27–30].

Suppose the zeros of $Q(t)$, the "turning points" (TPs), are first order. Since this is an over-the-barrier scattering problem, the TPs lie off the real axis, in the complex plane, and for real laser pulses they occur in complex conjugate pairs; furthermore, those closest to the real axis tend to dominate in the semiclassical regime. For a simple single-pulse field such as $E(t) = E\,\mathrm{sech}^2(\omega t)$, with $A(t) = -E/\omega \tanh(\omega t)$, or $E(t) = Ee^{-(\omega t)^2}$, with $A(t) = -\sqrt{\pi}E/(2\omega)\,\mathrm{erf}(\omega t)$, a single pair of complex conjugate TPs dominates, and the phase integral method leads to the familiar formula [19,27,30–32]

$$N(\vec{p}) \approx e^{-2K}, \qquad K = \left|\int_{\mathrm{TP}} Q dt\right|, \qquad (4)$$

where the integral is along the line joining the two complex conjugate TPs. In the ITM, one expands in momenta to obtain the general Gaussian expression [18]

$$N(\vec{p}) \approx \exp[-S_{\mathrm{cl}} - c_1 p_\perp^2 - c_2 p^2], \qquad (5)$$

where S_{cl} is the classical action evaluated on the contour, $c_1 = \frac{\partial}{\partial m^2} S_{\mathrm{cl}}$, and $c_2 = -2m^2 \frac{\partial^2}{\partial (m^2)^2} S_{\mathrm{cl}}$. This ITM result (5) [18,19] gives a compact expression that is the basis for most studies of vacuum pair production, and when integrated over momentum to give the total rate it gives excellent agreement with numerical (or exact) results [3,18,20,21,33,34]. In [35], (5) was applied to the envelope field (1) with $\phi = 0$. However, the expression (5) clearly cannot exhibit any of the numerically observed oscillations, as noted in [16]. This deficiency is not cured by higher-order terms in the momentum expansion in (5), nor is it cured by including higher-order WKB terms. The problem is that (4) and (5) are based on the assumption of just one pair of turning points, on the imaginary axis. However, for complicated fields, with subcycle structure, the essential shape of the "scattering potential," $-[p - A(t)]^2$, changes dramatically as p varies, as illustrated in Fig. 1. This can lead to scattering resonances, encoded semiclassically in *multiple* pairs of complex conjugate TPs, whose locations are correspondingly sensitive to variation of p. We show below that the oscillatory momentum behavior can be identified with interference effects between such pairs of turning points.

To illustrate the oscillatory phenomenon most clearly, we consider a simple analytic profile field that exhibits the effect found in [16]. The maximal oscillation effect occurs with carrier phase $\phi = \pi/2$, so that $E(t)$ is an odd function, and $A(t)$ an even function, so we choose

$$A(t) = E/[\omega(1 + \omega^2 t^2)]. \qquad (6)$$

The scattering potential is plotted in Fig. 1. The algebraic form of the vector potential makes it easy to find the turning points, occurring as two complex conjugate pairs: $t_1(p) = -[\sqrt{E/\omega - (p + im)}/\omega\sqrt{p + im}] = t_2(p)^* = -t_3(p)^* = -t_4(p)$, as shown in Fig. 2. Notice that all four TPs are equidistant from the real axis, for all p. Figure 2 also shows the Stokes and anti-Stokes lines. Since there are two pairs of TPs, the WKB analysis leading to (4) and (5) must be generalized to account for the crossing of multiple Stokes and anti-Stokes lines for multiple pairs of TPs in evolving from $t = -\infty$ to $t = +\infty$. This corresponds to the

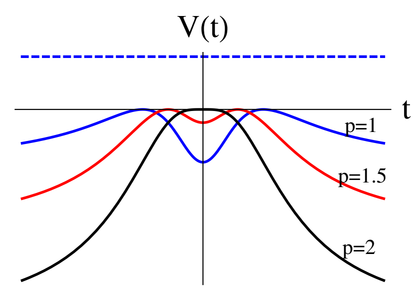

FIG. 1 (color online). The effective scattering potential, $V(t) = -[p - A(t)]^2$, for scalar QED, with the field in (6), for three different values of the longitudinal momentum, in units of electron mass m. Note that the form of the potential is highly sensitive to the value of p. The dashed line denotes the mass level m^2.

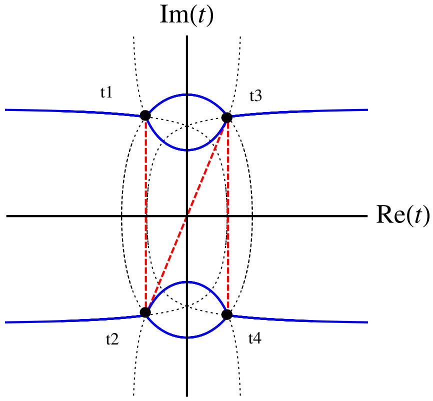

FIG. 2 (color online). The four complex turning points, $t_1, \ldots, t_4$, for the field (6), showing also the anti-Stokes lines (solid blue lines), Stokes lines (dotted black lines), and the integration contours (dashed red lines) used in (7) and (8).

case of over-the-barrier scattering with two bumps in the scattering potential, which has been solved in [36] using the phase integral approximation. Adapting their result, we find the simple expression

$$N_{\text{scalar}} \approx e^{-2K_1} + e^{-2K_2} + 2\cos(2\alpha)e^{-K_1-K_2},$$
$$K_1 = \left|\int_{t_1}^{t_2} Q dt\right|, \qquad K_2 = \left|\int_{t_3}^{t_4} Q dt\right|,$$
$$\alpha = L - \sigma(K_1) - \sigma(K_2), \qquad L = \left|\mathcal{R}e\left(\int_{t_2}^{t_3} Q dt\right)\right|,$$
$$\sigma(K) = \frac{1}{2}\left\{\frac{K}{\pi}\left[\ln\left(\frac{K}{\pi}\right) - 1\right] + \arg\Gamma\left(\frac{1}{2} - i\frac{K}{\pi}\right)\right\}. \qquad (7)$$

In fact, in this case $K_1 = K_2$, so we can write $N_{\text{scalar}} \approx 4\cos^2(\alpha)e^{-2K_1}$. Note the appearance in (7) of the interference term, $\cos(2\alpha)$, involving an integral between different pairs of turning points. This term is responsible for the oscillations in the momentum spectrum, as shown in Fig. 3 where we compare (7) with the exact numerical result, and with a naive application of the ITM result (4), just taking the first two terms in (7). The agreement of (7) with the numerical result is excellent. A generalization to more than two pairs of TPs is discussed in [37].

Having given a quantitative semiclassical explanation of the longitudinal momentum oscillations for scalar QED, we now turn to spinor QED, for which there is a similar scattering formulation [18,20,26]. The key difference is that the unitarity conditions on the spinor Bogoliubov coefficients have a reversed sign relative to the scalar case: now $|\alpha|^2 + |\beta|^2 = 1$. This changes the form of the F matrix of the phase integral approximation in [36], and is ultimately related to the double-valuedness of the spinor wave function. We find the scalar result (7) is modified to

$$N_{\text{spinor}} \approx e^{-2K_1} + e^{-2K_2} - 2\cos(2\alpha)e^{-K_1-K_2}, \qquad (8)$$

where the K_i and α are defined as in (7). The only change is the sign of the interference term. [When $K_1 = K_2$, as for the field in (6), we have $N_{\text{spinor}} \approx 4\sin^2(\alpha)e^{-2K_1}$.] Physically, this term is produced by interference between waves reflected by the double-bump structure, and for fermions there is an additional phase shift on reflection, which ultimately leads to this sign change. In Fig. 3 we plot this spinor result (8) and we see that it is in excellent agreement with the exact numerical results. The results (7) and (8) explain clearly why the oscillations are out of phase between spinor and scalar QED, and why the envelope of the two is the naive ITM result. Of course, if one is interested only in the *total* pair production rate, obtained by integrating over p, then the difference between spinor and scalar QED is washed out, and agrees with the answer obtained by integrating over the envelope result coming from just the first two terms in (7) or (8), since they oscillate about the same envelope. To conclude, we sketch in Fig. 4 the turning points of the carrier-phase field in (1),

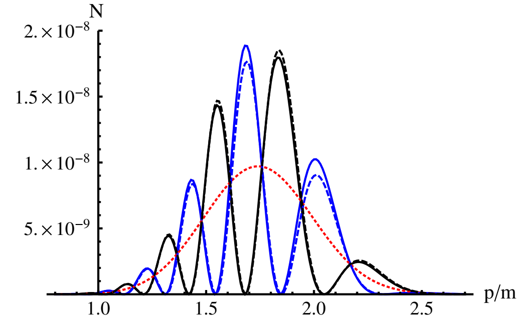

FIG. 3 (color online). Longitudinal momentum spectrum of e^+e^- pairs for the field (6), for $E = 0.2$, $\omega = 0.1$, all in units of electron mass m. The solid lines are our WKB expressions in (7) and (8), the dashed lines are exact numerical results, and the dotted (red) line is the naive ITM expression, neglecting the interference term. The oscillatory blue lines are scalar QED and the oscillatory black lines are spinor QED. The quantitative agreement of (7) and (8) with the numerics is excellent.

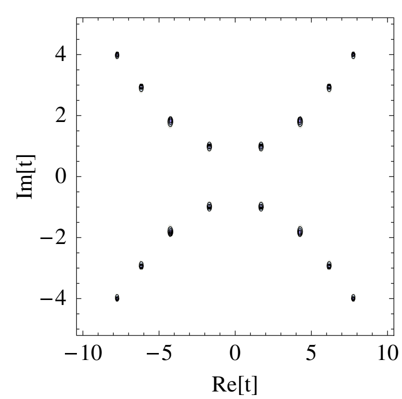

FIG. 4 (color online). Contour plot of $|m^2 + [p - A(t)]^2|$ for the electric field (1), with $\phi = \pi/2$, $p = 0$, and $\omega\tau = 4$, showing the infinite set of pairs of complex turning points. The two central pairs closest to the real axis dominate the semiclassical analysis.

for the case $\phi = \pi/2$. Note that there are infinitely many pairs of complex conjugate TPs. But the two pairs of TPs closest to the real axis dominate, and using formulas (7) and (8), we reproduce the oscillatory behavior of the electron-positron longitudinal momentum spectrum found numerically in [16]. When $\phi = 0$ the sensitivity to the value of $\omega\tau$ can also be understood in terms of the location of the TPs.

Our result is general and simple to use, and has applications beyond this particle-production context, for example, to strong-field ionization of atoms and molecules [13,19], to particle-production problems in cosmology [12,38], and to quasinormal modes of black holes [39]. The basic message is that when the time dependence of the external background field has more substructure than just a single bump, the usual textbook ITM result (4) generalizes in an interesting way that requires fuller consideration of the Stokes line structure of the associated scattering problem, and there are important differences between the momentum spectra of produced spinor or scalar particles. This also has important implications for the worldline approach to pair production [40], which has the potential to describe pair production in electric fields with both temporal and spatial inhomogeneity.

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