Gisin's Theorem for Arbitrary Dimensional Multipartite States

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We present a set of Bell inequalities which are sufficient and necessary for separability of general pure multipartite quantum states in arbitrary dimensions. The relations between Bell inequalities and distillability are also studied. We show that any quantum states that violate one of these Bell inequalities are distillable.

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Introduction.—One of the most remarkable aspects of quantum theory is the incompatibility of quantum nonlocality with local-realistic theories. The Bell inequalities [1] impose constraints on the correlations between measurement outcomes on two separated systems, giving rise to the limits for what can be described within the framework of any local hidden variable theory. They are of great importance for understanding the conceptual foundations of quantum theory as well as for investigating quantum entanglement, as Bell inequalities can be violated by quantum entangled states. One of the most important Bell inequalities is the Clauser-Horne-Shimony-Holt (CHSH) inequality [2] for two-qubit systems. It is then generalized to the N-qubit case, known as the Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality [3]. A set of multipartite Bell inequalities has been elegantly derived in terms of two dichotomic observables per site [4], which includes the MABK inequality as a special case [5] and can detect some entangled states that the MABK inequality fails to detect. In [6] another family of Bell inequalities for N-qubit systems has been introduced, which is maximally violated by all the Greenberger-Horne-Zeilinger states.

In fact, Gisin presented a theorem in 1991. It says that any pure entangled two-qubit states violate the CHSH inequality [7]. Namely the CHSH inequality is both sufficient and necessary for separability of two-qubit states. Soon after, Gisin and Peres provided an elegant proof of this theorem for the case of pure two-qudit systems [8]. In [9] Chen *et al.* showed that all pure entangled three-qubit states violate a Bell inequality. Nevertheless generally it still remains open whether the Gisin's theorem can be generalized to the N-qudit case or not.

The Bell inequalities are also useful in verifying the security of quantum key distribution protocols [10]. There is a simple relation between nonlocality and distillability: if any two-qubit [11] or three-qubit [12] pure or mixed state violates a specific Bell inequality, then the state must be distillable. In [13] Dür has shown that for the case $N \geq 8$, there exist N-qubit bound entangled (not distillable) states which violate some Bell inequalities. However, Acín has demonstrated that for all states violating the inequality, there exists at least one kind of bipartite decomposition of the system such that a pure entangled state can be distilled [14,15]. But generally it is still an open problem if violation of a Bell inequality already implies distillability.

In this paper, we present a set of Bell inequalities which can be shown to be both sufficient and necessary for separability of general pure multipartite quantum states in arbitrary dimensions, thus proving the Gisin's theorem generally. We also show that pure entangled states can be distilled from quantum mixed states that violate one of these Bell inequalities.

Bell inequalities for bipartite quantum systems.—For two-qubit quantum systems, the Bell operators are defined by

$$\mathcal{B} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2, \qquad (1)$$

where $A_i = \vec{a}_i \cdot \vec{\sigma}_A = a_i^x \sigma_A^x + a_i^y \sigma_A^y + a_i^z \sigma_A^z$, $B_j = \vec{b}_j \cdot \vec{\sigma}_B = b_j^x \sigma_B^x + b_j^y \sigma_B^y + b_j^z \sigma_B^z$, $\vec{a}_i = (a_i^x, a_i^y, a_i^z)$, and $\vec{b}_j = (b_j^x, b_j^y, b_j^z)$ are real unit vectors satisfying $|\vec{a}_i| = |\vec{b}_j| = 1$, $i, j = 1, 2$, $\sigma_{A/B}^{x,y,z}$ are Pauli matrices. The CHSH inequality says that if there exist local hidden variable models to describe the system, the inequality

$$|\langle \mathcal{B} \rangle| \leq 2 \qquad (2)$$

must hold.

Instead of a two-qubit (2×2) system, we first consider general $N \times M$ bipartite quantum systems in vector space $\mathcal{H}_{\mathcal{AB}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ with dimensions dim $\mathcal{H}_{\mathcal{A}} = M$ and dim $\mathcal{H}_{\mathcal{B}} = N$ respectively. We aim to find Bell inequalities like (2) such that any quantum entangled states would violate a Bell inequality.

Let L_α^A and L_β^B be the generators of special unitary groups $SO(M)$ and $SO(N)$, respectively. The $M(M-1)/2$ generators L_α^A are given by $\{|j\rangle\langle k| - |k\rangle\langle j|\}$, $1 \leq j < k \leq M$, where $|i\rangle$, $i = 1, \ldots, M$, are the usual orthonormal basis of $\mathcal{H}_{\mathcal{A}}$. L_β^B are similarly defined. The matrix operators L_α (respectively, L_β) have $M-2$ (respectively, $N-2$) rows and $M-2$ (respectively, $N-2$) columns that are identically zero. We define the operators A_i^α (respectively, B_j^β) from L_α (respectively, L_β) by replacing the four entries on the positions of the nonzero two rows and two columns of L_α (respectively, L_β) with the corresponding four entries of the matrix $\vec{a}_i \cdot \vec{\sigma}$ (respectively, $\vec{b}_j \cdot \vec{\sigma}$), and keeping the other entries of A_i^α (respectively, B_j^β) zero. We define the Bell operators to be

$$\mathcal{B}_{\alpha\beta} = \tilde{A}_1^\alpha \otimes \tilde{B}_1^\beta + \tilde{A}_1^\alpha \otimes \tilde{B}_2^\beta + \tilde{A}_2^\alpha \otimes \tilde{B}_1^\beta - \tilde{A}_2^\alpha \otimes \tilde{B}_2^\beta, \quad (3)$$

where $\tilde{A}_i^\alpha = L_\alpha A_i^\alpha L_\alpha^\dagger$, $\tilde{B}_j^\beta = L_\beta B_j^\beta L_\beta^\dagger$, and $i, j = 1, 2$.

Theorem 1.—Any bipartite pure quantum state is entangled if and only if at least one of the following Bell inequalities is violated:

$$|\langle \mathcal{B}_{\alpha\beta} \rangle| \le 2, \quad (4)$$

where $\alpha = 1, 2, \cdots, \frac{M(M-1)}{2}$, $\beta = 1, 2, \cdots, \frac{N(N-1)}{2}$.

Proof.—Assume that the state $|\psi\rangle$ violates one of the Bell inequalities in (4); i.e., there exist α_0 and β_0 such that $|\langle \mathcal{B}_{\alpha_0\beta_0} \rangle| > 2$. Then equivalently one has that the state $|\psi\rangle_{\alpha_0\beta_0} = \frac{L_{\alpha_0}^A \otimes L_{\beta_0}^B |\psi\rangle}{\| L_{\alpha_0}^A \otimes L_{\beta_0}^B |\psi\rangle \|}$ violates the CHSH inequality in (2). As the local operation $L_{\alpha_0}^A \otimes L_{\beta_0}^B$ does not change the separability of a state, $|\psi\rangle$ must be entangled.

Now assume that $|\psi\rangle \in \mathcal{H}_{\mathcal{AB}}$ is an entangled state. We prove that at least one of the Bell inequalities in (4) is violated. Set $\rho = |\psi\rangle\langle\psi|$. By projecting $|\psi\rangle$ onto 2×2 subsystems [16], we get the following pure states:

$$\rho_{\alpha\beta} = \frac{L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger}{\| L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger \|}, \quad (5)$$

where $\alpha = 1, 2, \cdots, \frac{M(M-1)}{2}$; $\beta = 1, 2, \cdots, \frac{N(N-1)}{2}$; and $\|X\| = \sqrt{\mathrm{Tr}(XX^\dagger)}$. Here $\rho_{\alpha\beta}$ are pure states with rank one. As the matrix $L_\alpha^A \otimes L_\beta^B$ has $MN - 4$ rows and $MN - 4$ columns that are identically zero, there are at most $4 \times 4 = 16$ nonzero elements in the matrix $\rho_{\alpha\beta}$. The states $\rho_{\alpha\beta}$ are called "two-qubit" states in this sense.

The concurrence of $|\psi\rangle$ is defined by $C(|\psi\rangle) = \sqrt{2[1 - \mathrm{Tr}(\rho_A^2)]}$ with $\rho_A = \mathrm{Tr}_B(\rho)$ the reduced density matrix of ρ by tracing over the subsystem B [17]. A pure quantum state $|\psi\rangle$ can be generally expressed as $|\psi\rangle = \sum_{i=1}^M \sum_{j=1}^N a_{ij}|ij\rangle$, $a_{ij} \in \mathbb{C}$, in the computational basis $|i\rangle$ and $|j\rangle$ of $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, respectively, $i = 1, \ldots, M$ and $j = 1, \ldots, N$. Therefore the concurrence can be expressed as

$$C(|\psi\rangle) = \sqrt{\sum_{\alpha=1}^{M} \sum_{\beta=1}^{N} |C(\rho_{\alpha\beta})|^2}, \quad (6)$$

where $\rho_{\alpha\beta}$ are defined in (5). Since we have assumed that $|\psi\rangle$ is an entangled quantum state, $C(|\psi\rangle)$ must be not zero; i.e., at least one of the $\rho_{\alpha\beta}$, say $\rho_{\alpha_0\beta_0}$, has nonzero concurrence, $C(\rho_{\alpha_0\beta_0}) > 0$. As we have discussed above, $\rho_{\alpha_0\beta_0}$ is actually a "two-qubit" quantum pure state. It has been shown in [7,8] that an entangled two-qubit pure state must violate the Bell inequality (2). Therefore the inequality $|\langle \mathcal{B}_{\alpha_0\beta_0} \rangle| \le 2$ is violated.

Bell inequalities for multipartite quantum systems.—We now generalize the results above to multipartite quantum systems. For convenience we consider that all the subsystems have the same dimensions. However, as can be seen from the following, our discussions also apply to multipartite quantum systems with different dimensions.

Let H denote a d-dimensional vector space with basis $|i\rangle$, $i = 1, 2, \ldots, d$. An L-partite pure state in $H \otimes \cdots \otimes H$ is generally of the form

$$|\Psi\rangle = \sum_{i_1, i_2, \cdots i_L = 1}^{d} a_{i_1, i_2, \cdots i_L} |i_1, i_2, \cdots i_N\rangle, a_{i_1, i_2, \cdots i_L} \in \mathbb{C}. \quad (7)$$

Let α and α' (respectively, β and β') be subsets of the subindices of a, associated to the same subvector spaces but with different summing indices. α (or α') and β (or β') span the whole space of the given subindix of a. A possible combination of the indices of α and β can be equivalently understood as a kind of bipartite decomposition of the L subsystems, say part A and part B, containing m and $n = L - m$ subsystems, respectively.

For a given bipartite decomposition, we can use the analysis similar to the bipartite case. Let L_α^A and L_β^B be the generators of special unitary groups $SO(d^m)$ and $SO(d^n)$. By projecting $|\Psi\rangle$ onto 2×2 subsystems we have the "two-qubit" pure states:

$$\rho_{\alpha\beta}^p = \frac{L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger}{\| L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger \|}, \quad (8)$$

where $\alpha = 1, 2, \cdots, \frac{d^m(d^m-1)}{2}$; $\beta = 1, 2, \cdots, \frac{d^n(d^n-1)}{2}$, p labels the bipartite decompositions of the L subsystems.

For every pure state $\rho_{\alpha\beta}^p$ we define the corresponding Bell operators

$$\mathcal{B}_{\alpha\beta}^p = \tilde{A}_1^\alpha \otimes \tilde{B}_1^\beta + \tilde{A}_1^\alpha \otimes \tilde{B}_2^\beta + \tilde{A}_2^\alpha \otimes \tilde{B}_1^\beta - \tilde{A}_2^\alpha \otimes \tilde{B}_2^\beta, \quad (9)$$

where $\tilde{A}_i^\alpha = L_\alpha^A A_i^\alpha (L_\alpha^A)^\dagger$ and $\tilde{B}_j^\beta = L_\beta^B B_j^\beta (L_\beta^B)^\dagger$ are the Hermitian operators similarly defined as in (3).

Theorem 2.—Any multipartite pure quantum state is entangled if and only if at least one of the following inequalities is violated:

$$|\langle \mathcal{B}_{\alpha\beta}^p \rangle| \le 2. \quad (10)$$

Proof.—Obviously, multipartite quantum states that violate any one of the Bell inequalities in (10) must be entangled.

We now prove that, for any entangled multipartite pure quantum state, at least one of the inequalities in (10) is violated. The concurrence of $|\Psi\rangle$ is given by [18]

$$C_d^L(|\Psi\rangle) = \sqrt{K\sum_p \sum_{\{\alpha,\alpha',\beta,\beta'\}}^{d} |a_{\alpha\beta}a_{\alpha'\beta'} - a_{\alpha\beta'}a_{\alpha'\beta}|^2}, \quad (11)$$

where $K = d/2m(d-1)$, $m = 2^{L-1} - 1$, $\sum_p$ stands for the summation over all possible combinations of the indices of α and β. (11) can be rewritten as

$$C_d^L(|\Psi\rangle) = \sqrt{K\sum_p \sum_{\alpha\beta}[C(\rho_{\alpha\beta}^p)]^2}, \quad (12)$$

where $\rho_{\alpha\beta}^p$ are defined in (8). As $|\Psi\rangle$ is an entangled state, $C(|\Psi\rangle)$ must be not zero; i.e., at least one of $\rho_{\alpha\beta}^p$, say $\rho_{\alpha_0\beta_0}^{p_0}$, has nonzero concurrence. As we have discussed above, $\rho_{\alpha_0\beta_0}^{p_0}$ is actually a two-qubit quantum pure state. An entangled two-qubit quantum pure state must violate the Bell inequality (2).

As an example, we consider three-qubit systems. In [19], Acin, etc., have verified that any pure three-qubit state $|\Psi\rangle$ can be uniquely written as

$$|\Psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\psi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \quad (13)$$

where $\lambda_i \geq 0$, $0 \leq \psi \leq \pi$, $\sum_i \lambda_i^2 = 1$. From straightforward computation one has

$$\begin{aligned} C^2(|\Psi\rangle) = &2(\lambda_0\lambda_2)^2 + 2(\lambda_0\lambda_4)^2 + |2e^{i\psi}\lambda_1\lambda_4 - 2\lambda_2\lambda_3|^2 \\ &+ 2(\lambda_0\lambda_3)^2 + 2(\lambda_0\lambda_4)^2 + |2e^{i\psi}\lambda_1\lambda_4 - 2\lambda_2\lambda_3|^2 \\ &+ 2(\lambda_0\lambda_2)^2 + 2(\lambda_0\lambda_3)^2 + 2(\lambda_0\lambda_4)^2. \end{aligned}$$

We give a detailed analysis on that an entangled pure three-qubit state, i.e., at least one of the terms in the right-hand side of (14) is nonzero, must violate one of the inequalities in (10).

Case 1.—If $\lambda_0\lambda_2 \neq 0$, the corresponding operator

$$L_2^A \otimes L_1^B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$\rho_{21}^{12|3} = \begin{pmatrix} \lambda_2^2 & -e^{-i\psi}\lambda_1\lambda_2 & 0 & 0 & 0 & \lambda_0\lambda_2 & 0 & 0 \\ -e^{i\psi}\lambda_1\lambda_2 & \lambda_1^2 & 0 & 0 & 0 & -e^{i\psi}\lambda_0\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_0\lambda_2 & -e^{-i\psi}\lambda_0\lambda_1 & 0 & 0 & 0 & \lambda_0^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Choose the Bell operator in (9) to be the one with respect to the bipartite decomposition of the first two qubits and the last one,

$$\mathcal{B}_{21}^{12|3} = \tilde{A}_1^2 \otimes \tilde{B}_1^1 + \tilde{A}_1^2 \otimes \tilde{B}_2^1 + \tilde{A}_2^2 \otimes \tilde{B}_1^1 - \tilde{A}_2^2 \otimes \tilde{B}_2^1, \quad (14)$$

where $\tilde{A}_k^2 = L_2^A A_k^2 (L_2^A)^\dagger$, $\tilde{B}_l^1 = L_1^B B_l^1 (L_1^B)^\dagger$, and

$$A_k^2 = \begin{pmatrix} -a_k^3 & 0 & a_k^1 + a_k^2 i & 0 \\ 0 & 0 & 0 & 0 \\ a_k^1 - a_k^2 i & 0 & a_k^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_l^1 = \begin{pmatrix} -b_l^3 & b_l^1 + b_l^2 i \\ b_l^1 - b_l^2 i & b_l^3 \end{pmatrix},$$

$k, l = 1, 2$; we have the maximal violation of the inequality (10), $2\sqrt{1 + 4\lambda_0^2\lambda_2^2/(\lambda_0^2 + \lambda_1^2 + \lambda_2^2)^2} > 2$.

Case 2.—If $|e^{i\psi}\lambda_1\lambda_4 - \lambda_2\lambda_3| \neq 0$, the corresponding operator

$$L_6^A \otimes L_1^B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The matrix $\rho_{61}^{12|3}$ has only nonzero entries at the right down corner with the form,

$$\begin{pmatrix} \lambda_4^2 & -\lambda_3\lambda_4 & -\lambda_2\lambda_4 & e^{-i\psi}\lambda_1\lambda_4 \\ -\lambda_3\lambda_4 & \lambda_3^2 & \lambda_2\lambda_3 & -e^{-i\psi}\lambda_1\lambda_3 \\ -\lambda_2\lambda_4 & \lambda_2\lambda_3 & \lambda_2^2 & -e^{-i\psi}\lambda_1\lambda_2 \\ e^{i\psi}\lambda_1\lambda_4 & -e^{i\psi}\lambda_1\lambda_3 & -e^{i\psi}\lambda_1\lambda_2 & \lambda_1^2 \end{pmatrix}.$$

The Bell operator in (9) has the form,

$$\mathcal{B}_{61}^{12|3} = \tilde{A}_1^6 \otimes \tilde{B}_1^1 + \tilde{A}_1^6 \otimes \tilde{B}_2^1 + \tilde{A}_2^6 \otimes \tilde{B}_1^1 - \tilde{A}_2^6 \otimes \tilde{B}_2^1, \quad (15)$$

where $\tilde{A}_k^6 = L_6^A A_k^6 (L_6^A)^\dagger$, $\tilde{B}_l^1 = L_1^B B_l^1 (L_1^B)^\dagger$, and

$$A_k^6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a_k^3 & a_k^1 + a_k^2 i \\ 0 & 0 & a_k^1 - a_k^2 i & a_k^3 \end{pmatrix},$$

$$B_l^1 = \begin{pmatrix} -b_l^3 & b_l^1 + b_l^2 i \\ b_l^1 - b_l^2 i & b_l^3 \end{pmatrix},$$

$k, l = 1, 2$. The corresponding maximal violation is given by $2\sqrt{1 + 4|e^{i\psi}\lambda_1\lambda_4 - \lambda_2\lambda_3|^2/(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)^2}$, which is obviously strictly larger than 2. Other cases can be discussed similarly.

Bell inequalities and distillation.—A bipartite state ρ is called distillable, if maximally entangled bipartite pure states, e.g., $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, can be created from a number of identical copies of the state ρ by means of local operations and classical communication. We call a multipartite state distillable, if and only if there exists at least one bipartite decomposition of the system such that pure entangled states can be distilled. It has been shown that all quantum entangled pure states are distillable. However it is a challenge to give an operational criterion of distillability for general mixed states. In [16] a sufficient condition of distillability has been presented. Our inequalities (10) are both sufficient and necessary for separability of pure states, but generally not for separability of mixed ones. However surprisingly (10) can be served as criterion for distillability.

Theorem 3.—Any bipartite quantum state ρ that violates any one of the Bell inequalities in (4), i.e. $\mathrm{Tr}\{\mathcal{B}_{\alpha\beta}\rho\} > 2$, is always distillable. And if a multipartite quantum state ρ violates one of the Bell inequalities in (10), i.e., ρ satisfies $\mathrm{Tr}\{\mathcal{B}^{p}_{\alpha\beta}\rho\} > 2$, then bipartite maximally entangled pure states can be distilled from the copies of ρ.

Proof.—It was shown in [20] that a density matrix ρ is distillable if there are some projectors P, Q that map high dimensional spaces to two-dimensional ones such that the state $P \otimes Q\rho^{\otimes s}P \otimes Q$ is entangled for some s copies. Thus if any one of the Bell inequalities in (4) is violated, there exists a submatrix $\rho_{\alpha\beta}$, like (5), that has nonzero concurrence. For generally given operator $L_\alpha = |i\rangle\langle j| - |j\rangle\langle i|$, $L_\beta = |k\rangle\langle l| - |l\rangle\langle k|$, the operators P, Q can be explicitly given by $P = AL_\alpha$, $Q = BL_\beta$, where $A = |0_A\rangle\langle i| + |1_A\rangle\langle j|$, $B = |0_B\rangle\langle k| + |1_B\rangle\langle l|$, $|0_{A/B}\rangle$, and $|1_{A/B}\rangle$ are the orthonormal bases of a two-dimensional vector space. $P \otimes Q$ maps state ρ to a two-qubit one that has the same nonzero concurrence as $\rho_{\alpha\beta}$. Since any entangled two-qubit state is distillable, ρ is distillable. The multipartite case can be discussed similarly.

Remark.—It has been shown thatpositive partial transposition (PPT) entangled quantum states are not distillable [21]. Therefore PPT quantum states should never violate the Bell inequalities in (4) or (10). This fact can be seen from the following. A density matrix ρ is called PPT if the partial transposition of ρ with respect to any subsystem(s) is still positive. Let ρ^{T_B} denote the partial transposition with respect to the subsystem B. Assume that there is a PPT state ρ violating one of the Bell inequalities in (10), say $\mathrm{Tr}\{\mathcal{B}^{p_0}_{\alpha_0\beta_0}\rho\} > 2$. This can be equivalently understood as that there exists two-qubit state $\rho^{p_0}_{\alpha_0\beta_0}$ in the form of (8) such that $\mathrm{Tr}\{B^{p_0}_{\alpha_0\beta_0}\rho^{p_0}_{\alpha_0\beta_0}\} > 2$, where $B^{p_0}_{\alpha_0\beta_0} = A^{\alpha_0}_1 \otimes B^{\beta_0}_1 + A^{\alpha_0}_1 \otimes B^{\beta_0}_2 + A^{\alpha_0}_2 \otimes B^{\beta_0}_1 - A^{\alpha_0}_2 \otimes B^{\beta_0}_2$. One the other hand, by using the PPT property of ρ, we have

$$\rho^{T_B}_{\alpha_0\beta_0} = L^A_{\alpha_0} \otimes (L^B_{\beta_0})^* \rho^{T_B} (L^A_{\alpha_0})^\dagger \otimes (L^B_{\beta_0})^T \geq 0. \quad (16)$$

As both $L^A_{\alpha_0}$ and $L^B_{\beta_0}$ are projectors to two-dimensional subspaces, $\rho^{p_0}_{\alpha_0\beta_0}$ can be considered as a 2×2 state. While a 2×2 PPT state $\rho_{\alpha_0\beta_0}$ must be separable [22], it contradicts with $\mathrm{Tr}\{B^{p_0}_{\alpha_0\beta_0}\rho^{p_0}_{\alpha_0\beta_0}\} > 2$.

Conclusions and remarks.—In conclusion, we have derived a series of new Bell inequalities for both bipartite and multipartite quantum states by projecting the whole quantum systems to "two-qubit" subsystems. We show that quantum states violating any one of these Bell inequalities are entangled. On the other hand, we have proved that any entangled pure quantum states must violate at least one of these Bell inequalities. Thus the Gisin theorem for general multipartite quantum systems has been proved. We have also shown that quantum states that violate the Bell inequalities must be distillable, which helps on measurable determination of quantum entanglement experimentally.

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