Short-Time Operator Product Expansion for rf Spectroscopy of a Strongly Interacting Fermi Gas

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Universal relations that hold for any state provide powerful constraints on systems consisting of fermions with two spin states interacting with a large scattering length. In radio-frequency (rf) spectroscopy, the mean shift in the rf frequency and the large-frequency tail of the rf transition rate are proportional to the contact, which measures the density of pairs with small separations. We show that these universal relations can be derived and extended by using the short-time operator product expansion of quantum field theory. This is a general method for identifying aspects of many-body physics that are controlled by few-body physics.

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Trapped ultracold atoms allow the study of the few-body physics and the many-body physics of systems in which the fundamental interactions between the constituents are understood and can be controlled experimentally. The possibility of making the interactions between the atoms arbitrarily strong presents a challenge to theory, because many theoretical methods break down when interactions become too strong. Specifying the interactions between the constituents of the many-body system in terms of two-body interactions is a trivial example of a connection between few-body physics and many-body physics, but there can be deeper connections. Among the simplest examples of strongly interacting systems are ones that consist of atoms that interact with a large scattering length [1]. The two-body problem for such atoms can be solved analytically, the three-body problem can be solved exactly numerically, and the four-body problem is becoming tractable using modern computers. This makes it an interesting system for studying how nontrivial aspects of many-body physics can be controlled by few-body physics.

A powerful experimental tool for studying ultracold atoms is radio-frequency (rf) spectroscopy, in which atoms in one hyperfine spin state are excited into a different spin state. Pioneering applications of this method were measurements of the binding energy of weakly bound diatomic molecules of ^{40}K atoms [2] and the study of the pairing gap in a many-body system of ^{6}Li atoms [3]. The method had been extended to allow the spacial resolution of trapped systems [4] and the momentum resolution of the excited atoms [5]. A recent review of rf spectroscopy has explored the relation to photoemission experiments on high-temperature superconductors [6].

Universal relations between various properties of a system that must be satisfied in any state can provide powerful constraints on theoretical methods. Shina Tan has derived universal relations for systems consisting of fermions with two spin states that interact with a large scattering length a [7]. These relations all involve a property of the system called the *contact*. It is an extensive quantity that can be expressed as the integral over space of the *contact density*, which is proportional to the number of pairs with different spins per (volume)$^{4/3}$ [8]. The Tan relations include the coefficient of the $1/k^4$ tail in the momentum distributions at large momentum, a decomposition of the total energy into terms that are insensitive to short distances, the rate of change of the free energy due to a change in a, the relation between the pressure and the energy density in a homogeneous system, and the virial theorem for a system in a harmonic trapping potential [7]. Tan derived his relations within the framework of the many-body Schrödinger equation using novel methods involving generalized functions. In Ref. [8], Braaten and Platter rederived the Tan relations within the framework of quantum field theory using standard renormalization methods together with the short-distance operator product expansion. The Tan relations have also been rederived using less formal methods [9].

The contact also plays an important role in the rf spectroscopy of ultracold atoms. In a many-body system consisting of atoms in spin states 1 and 2, a rf signal with frequency ω can be used to transfer atoms from the spin state 2 into a third spin state 3. There are sum rules that constrain the rf transition rate $\Gamma(\omega)$ [10]. Of particular interest is the case of fermionic atoms for which every pair has strong interactions determined by large scattering lengths a_{12}, a_{13}, and a_{23}. An example is the lowest three hyperfine spin states of ^{6}Li atoms at generic values of the magnetic field. In this case, the sum rules are

$$\int_{-\infty}^{\infty} d\omega \Gamma(\omega) = \pi \Omega^2 N_2, \tag{1a}$$

$$\int_{-\infty}^{\infty} d\omega\, \omega \Gamma(\omega) = (\Omega^2/4m)(a_{12}^{-1} - a_{13}^{-1}) C_{12}, \tag{1b}$$

where Ω is the Rabi frequency determined by the strength of the rf signal, N_2 is the number of atoms of type 2, and C_{12} is the contact for atoms of types 1 and 2. We measure the rf frequency ω relative to the hyperfine frequency difference between atoms 2 and 3, and we set $\hbar = 1$. The sum rule in Eq. (1b) [11,12] implies that the mean shift in

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the rf frequency due to interactions is proportional to the contact and vanishes if $a_{13} = a_{12}$. The coefficient of the contact diverges in the limit $a_{13} \to 0$, which indicates that the mean shift in the rf frequency is sensitive to the range r_0 of the interactions between the atoms if a_{13} is not large. The sensitivity to the range arises in this case because $\Gamma(\omega)$ has a high-frequency tail that decreases like $\omega^{-3/2}$ and is proportional to the contact [13]:

$$\Gamma(\omega) \to \frac{\Omega^2}{4\pi\sqrt{m}\omega^{3/2}} C_{12} \qquad (a_{13} \approx 0). \tag{2}$$

The power-law tail extends out to ω of order $1/(mr_0^2)$, beyond which it is cut off by range effects.

In this Letter, we point out that the short-time operator product expansion of quantum field theory provides a general method for deriving connections between few-body physics and many-body physics. We use it to rederive and extend the universal results for rf spectroscopy in Eqs. (1b) and (2). We also use it to derive new sum rules that are insensitive to range effects.

The operator product expansion (OPE) is a powerful tool for studying strongly-interacting quantum field theories that was invented independently by Wilson, Kadanoff, and Polyakov in 1969 [14]. It expresses the product of local operators at different space-time points as an expansion in local operators with coefficients that are functions of the separation of the operators:

$$\mathcal{O}_A\left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}, T + \frac{1}{2}t\right) \times \mathcal{O}_B\left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}, T - \frac{1}{2}t\right) = \sum_C W_C(\boldsymbol{r}, t)\mathcal{O}_C(\boldsymbol{R}, T). \tag{3}$$

The sum is over the infinitely many local operators $\mathcal{O}_C$. The functions $W_C(\boldsymbol{r}, t)$ are called *Wilson coefficients*. The *short-distance* OPE, with the operators at the same time ($t = 0$), is an asymptotic expansion for small $|\boldsymbol{r}|$ [15]. It can be generalized to a *short-time* OPE by analytically continuing the time difference t to a Euclidean time τ defined by $t = -i\tau$. This form of the OPE is an asymptotic expansion for small $|\boldsymbol{r}|$ and τ.

A classic application of the short-time OPE in high-energy physics is electron-positron (e^+e^-) annihilation. In high-energy e^+e^- collisions, most of the annihilation cross section is into hadrons (mesons and baryons). The fundamental theory for hadrons is quantum chromodynamics (QCD), which is a strongly-interacting quantum field theory. As far as the hadrons are concerned, the annihilation of an e^+e^- pair with center-of-mass energy E results in the electromagnetic current operator $\boldsymbol{J}(\boldsymbol{r}, t)$ acting on the QCD vacuum state. This creates a quark and antiquark at a point with total energy E and with equal and opposite momenta. They are subsequently transformed by the strong interactions of QCD into one or more hadrons. The inclusive cross section $\sigma(E)$ for e^+e^- annihilation into hadrons can be expressed formally in terms of the expectation value in the QCD vacuum of a product of electromagnetic currents:

$$\sigma(E) = (4\pi\alpha/3E^4)\mathrm{Im}\, i \int dt e^{i(E+i\epsilon)t} \times \int d^3r \langle 0|T\boldsymbol{J}(\boldsymbol{r}, t) \cdot \boldsymbol{J}(0, 0)|0\rangle, \tag{4}$$

where $\alpha \approx 1/137$ is the fine structure constant of QED. The short-time OPE can be used to expand the operator product $\boldsymbol{J}(\boldsymbol{r}, t) \cdot \boldsymbol{J}(\boldsymbol{0}, 0)$ in powers of local operators with increasing dimensions. The leading operator is the unit operator, whose expectation value in any normalized state is 1 and whose dimension is 0. The next higher dimension operators that have nonzero expectation values in the QCD vacuum are scalar quark operators with dimension 3 and the gluon field strength operator with dimension 4. Because of the asymptotic freedom of QCD, the Wilson coefficients can be calculated using perturbation theory in the QCD coupling constant α_s. For large complex E, the Fourier transforms of the Wilson coefficients of the higher dimension operators are suppressed by powers of E. The cross section at asymptotically large energy E can therefore be obtained by truncating the OPE after the unit operator and calculating its Wilson coefficient to leading order in α_s. Inserting the OPE into Eq. (4) and neglecting quark masses relative to E, one obtains the simple result

$$\sigma(E) \to 4\pi\alpha^2 \sum_q e_q^2/E^2, \tag{5}$$

where the sum is over the quark flavors q and e_q is the electric charge ($+\frac{2}{3}$ or $-\frac{1}{3}$) of the quark [16]. The leading corrections at large E decrease as powers of $1/\ln(E)$ and can be calculated using perturbation theory in α_s.

To apply the OPE to ultracold atoms, the problem must be formulated in terms of a local quantum field theory. Fermionic atoms with hyperfine spin states 1, 2, and 3 that interact with large pair scattering lengths a_{12}, a_{23}, and a_{13} can be described by a local quantum field theory with quantum fields $\psi_\sigma(\boldsymbol{r})$, $\sigma = 1, 2, 3$. The interaction term in the Hamiltonian density that gives the large scattering length a_{12} is

$$\mathcal{H}_{\mathrm{int}} = (\lambda_{0,12}/m)\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1. \tag{6}$$

The bare coupling constant $\lambda_{0,12}$ depends on the ultraviolet momentum cutoff Λ: $\lambda_{0,12} = 4\pi/(1/a_{12} - 2\Lambda/\pi)$. Similar interaction terms give the large scattering lengths a_{13} and a_{23}. In Ref. [8], the contact C_{12} associated with atoms 1 and 2 was identified as the integral over space of the expectation value of a local operator:

$$C_{12} = \int d^3R \langle \lambda_{0,12}^2 \psi_1^\dagger\psi_2^\dagger\psi_2\psi_1(\boldsymbol{R})\rangle. \tag{7}$$

Note that the contact density operator has an additional factor of the bare coupling constant compared to the interaction term in Eq. (6). In Ref. [8], two of Tan's universal relations were rederived using the short-distance OPE. The

OPE for $\psi_\sigma^\dagger$ and ψ_σ was used to show that the momentum distributions for atoms of types 1 and 2 have identical $1/k^4$ tails whose coefficient is the contact C_{12}. The OPE for $\psi_1^\dagger\psi_1$ and $\psi_2^\dagger\psi_2$ was used to show that the contact density is proportional to the number of pairs of atoms of types 1 and 2 per (volume)$^{4/3}$.

The universal relations for rf spectroscopy in Eqs. (1b) and (2) can be rederived and extended by using the short-time OPE. We consider a many-body system consisting of ultracold atoms of types 1 and 2 only. As far as the atoms are concerned, rf spectroscopy with angular frequency ω proceeds through the action of the local operator $\psi_3^\dagger\psi_2(\boldsymbol{r}, t)$ on the many-body state. The operator transforms an atom of type 2 into an atom of type 3 with energy larger by ω. The inclusive rate $\Gamma(\omega)$ for producing final states containing an atom of type 3 can be expressed formally in terms of the expectation value in the many-body state of a product of operators:

$$\Gamma(\omega) = \Omega^2 \mathrm{Im} i \int dt e^{i(\omega+i\epsilon)t} \int d^3R \int d^3r \left\langle T \psi_2^\dagger \psi_3\left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}, t\right) \psi_3^\dagger \psi_2\left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}, 0\right)\right\rangle. \tag{8}$$

The Wilson coefficients in the OPE are determined by few-body physics and can be calculated diagrammatically using methods described in Ref. [17]. The lowest dimension operators that have nonzero expectation values in a state that contains no atoms of type 3 are $\psi_2^\dagger\psi_2$, which has scaling dimension 3, and the contact density operator $\lambda_{0,12}^2\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1$ which has scaling dimension 4. The scaling dimensions of these and other operators can be deduced from the operator-state correspondence of conformal field theory [18]. The Fourier transforms of the corresponding terms in the OPE are

$$\int dt e^{i\omega t} \int d^3r \psi_2^\dagger \psi_3\left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}, t\right) \psi_3^\dagger \psi_2\left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}, 0\right) = (i/\omega)\psi_2^\dagger\psi_2(\boldsymbol{R}) + iW_{12}(\omega)\lambda_{0,12}^2\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1(\boldsymbol{R}) + \ldots. \tag{9}$$

The Wilson coefficients can be determined by matching the matrix elements of both sides between asymptotic incoming and outgoing few-atom states. For $\psi_2^\dagger\psi_2$, one can use single-atom states of type 2. For the contact density operator, one can use two-atom scattering states consisting of atoms of types 1 and 2. The matrix elements for these states can be calculated using the analytic solution to the two-body problem. The matrix element of the contact density operator is proportional to $1/|1/a_{12} + ik|^2$, where k^2/m is the total energy of the scattering atoms. Its Wilson coefficient is

$$W_{12}(\omega) = \frac{a_{12}^{-1} - a_{13}^{-1}}{4\pi m\omega^2}\frac{a_{12}^{-1} - \sqrt{-m\omega}}{a_{13}^{-1} - \sqrt{-m\omega}}. \tag{10}$$

The OPE in Eq. (9) is an asymptotic expansion for large $|\omega|$ along any ray in the complex ω plane except possibly along the real ω axis, where the Wilson coefficients have poles and branch points.

The scaling of the Wilson coefficients at large ω is determined simply by the scaling dimensions of the operators and by symmetries. The rf transition operators on the left side of the OPE in Eq. (9) each have scaling dimension 3. Since the integral over nonrelativistic space and time has dimension -5, the total scaling dimension is 1. On the right side of the OPE, the Wilson coefficient of an operator of dimension d must have terms that scale as $\omega^{(1-d-n)/2}(a^{-1})^n$, where a is a scattering length. Unless factors of a^{-1} are required by a symmetry, the leading behavior is $\omega^{(1-d)/2}$. For $\psi_2^\dagger\psi_2$, which has scaling dimension 3, the leading behavior is correctly predicted to be ω^{-1}. The contact density operator has scaling dimension 4, so the leading behavior might be expected to be $\omega^{-3/2}$. However the rf transition operators are invariant under a subgroup of an $SU(2)$ symmetry that is respected by the Hamiltonian if $a_{12} = a_{13}$ [10]. The contact density operator does not respect this symmetry, so $W_{12}(\omega)$ must have a factor of $a_{13}^{-1} - a_{12}^{-1}$ that vanishes when $a_{12} = a_{13}$. Thus its leading behavior is ω^{-2}, in agreement with Eq. (10).

The short-time OPE can be used to derive sum rules for integrals over the rf frequency of the form $\int_{-\infty}^{\infty} d\omega F(\omega)\Gamma(\omega)$, where $F(\omega)$ is a weight function that is analytic on the real ω axis. Upon inserting the expression in Eq. (8) for $\Gamma(\omega)$ as a discontinuity in ω into the integral, it can be expressed as a line integral over a contour that wraps around the real axis. The sum rules in Eq. (1) can be derived by deforming the contour into a circle at infinity in the complex ω plane. The short-time OPE converges everywhere along the contour except possibly near the real axis. Inserting the OPE in Eq. (9) into the expression for $\Gamma(\omega)$ in Eq. (8) and then evaluating the contour integrals, we obtain the sum rules in Eq. (1).

The short-time OPE can also be used to derive the large-frequency behavior of the rf intensity. Inserting the OPE in Eq. (9) into the expression for $\Gamma(\omega)$ in Eq. (8), we obtain

$$\Gamma(\omega) \longrightarrow \frac{\Omega^2(a_{12}^{-1} - a_{13}^{-1})^2}{4\pi\sqrt{m}\omega^{3/2}(a_{13}^{-2} + m\omega)}C_{12}. \tag{11}$$

It was pointed out in Ref. [19] that $\Gamma(\omega)$ should decrease asymptotically like $\omega^{-5/2}$. The coefficient of the contact in the $\omega^{-5/2}$ tail is a new result. The analytic result for the rf transition rate $\Gamma(\omega)$ for the weakly-bound dimer associated with large positive a_{12} [20] agrees with Eq. (11) in the limit $\omega \gg 1/(ma_{12}^2)$ if we use the fact that the contact of the dimer is $C_{12} = 8\pi/a_{12}$ [7]. The result in Eq. (11) for the tail in the rf transition rate relies on the scattering length a_{13} being large compared to the range r_0. If a_{13} is not large, the corresponding result can be obtained by taking the limit $a_{13} \to 0$ in Eq. (11), which gives Eq. (2).

Sum rules that are less sensitive to range effects can be obtained by using a weight function $F(\omega)$ that decreases as $\omega \to \infty$. Using a Lorentzian centered at ω_0 with half-width at half-maximum γ gives a two-parameter family of sum rules. Upon deforming the contour into a circle at infinity along which the integral vanishes, we are left with the contributions from the poles in ω at $\omega_0 \pm i\gamma$. The convergence of the expansion for the sum rule is governed by the convergence of the OPE at this complex frequency and is therefore insensitive to the breakdown of the OPE near the real axis. In the limit $a_{13} \to 0$, this sum rule is

$$\int_{-\infty}^{\infty} d\omega \frac{\gamma/\pi}{(\omega-\omega_0)^2+\gamma^2}\Gamma(\omega) = \frac{\Omega^2\gamma}{\omega_\gamma^2}N_2 + \frac{\Omega^2[(\omega_0^2-\gamma^2)b_+ + 2\omega_0\gamma b_- - 2\omega_0\gamma a_{12}^{-1}]}{4\pi m\omega_\gamma^4}C_{12} + \ldots . \quad (12)$$

where $\omega_\gamma = (\omega_0^2+\gamma^2)^{1/2}$ and $b_\pm = [m(\omega_\gamma \pm \omega_0)/2]^{1/2}$. The analogous sum rule for e^+e^- annihilation into hadrons was derived in Ref. [21]. Sum rules of the form $\int_0^{\omega_0} d\omega F(\omega)\Gamma(\omega)$, where $F(\omega)$ is a polynomial, are also insensitive to range effects if $\omega_0 \ll 1/(mr_0^2)$. The convergence of the expansion for the sum rule is governed by the convergence of the OPE on the circle $|\omega| = \omega_0$. Sensitivity to the breakdown of the OPE near the real axis can be decreased by including a factor of $\omega - \omega_0$ in $F(\omega)$.

In the OPE in Eq. (9), there are additional terms that have nonzero matrix elements in states that contain atoms of type 3. The lowest dimension operator is the 23 analog of the 12 contact density operator in Eq. (7). If the rf operators in Eq. (9) are replaced by their Hermitian conjugates, the local operators include $\psi_3^\dagger\psi_3$ and the 13 contact density operator. There is also a 123 contact density operator proportional to $\psi_1^\dagger\psi_2^\dagger\psi_3^\dagger\psi_3\psi_2\psi_1$, whose scaling behavior is governed by Efimov physics [1]. It has a complex scaling dimension whose real part is 5 and whose imaginary part is $\pm 2s_0$, where $s_0 \approx 1.006\,24$. Thus its Wilson coefficient will decrease asymptotically like ω^{-2} multiplied by a log-periodic function of ω with a discrete scaling factor of approximately 515. The calculation of its Wilson coefficient requires the numerical solution to a three-body problem.

The short-time OPE can be applied to other operators to derive sum rules and high-frequency tails for the associated spectral functions. The OPE for $\psi_\sigma^\dagger$ and ψ_σ provides universal information about the spectrum of single-particle excitations. The OPE for density operators $\psi_\sigma^\dagger\psi_\sigma$ provides universal information about the spectrum of density fluctuations. The OPE for two stress tensors provides constraints on the spectral functions that determine transport coefficients, such as the viscosity.

Universal relations involving the contact provide nontrivial examples of aspects of many-body physics that are controlled by few-body physics. The OPE reveals these aspects by expressing observables in terms of Wilson coefficients that are determined by few-body physics. Of the known universal relations, most have been derived from the analytic solution to the two-body problem, but one has been derived from the solution to the three-body problem [22]. The derivation of new universal relations from numerical solutions to the three-body and higher few-body problems presents an interesting challenge.

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Note added.—The contact has recently been measured in experiments with ultracold ^{6}Li atoms [23] and ^{40}K atoms [24]. Several of the universal relations involving the contact have been verified experimentally.

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