Critical Fluctuations in Spatial Complex Networks

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An anomalous mean-field solution is known to capture the nontrivial phase diagram of the Ising model in annealed complex networks. Nevertheless, the critical fluctuations in random complex networks remain mean field. Here we show that a breakdown of this scenario can be obtained when complex networks are embedded in geometrical spaces. Through the analysis of the Ising model on annealed spatial networks, we reveal, in particular, the spectral properties of networks responsible for critical fluctuations and we generalize the Ginsburg criterion to complex topologies.

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A great deal of attention has been given recently to the effects that different topological properties may induce on the behavior of equilibrium and nonequilibrium processes defined on networks and to the possible implications for the study of several social, biological, and technological networks [1,2]. Heterogeneous degree distributions, small world and spectral properties, in particular, have been recognized as being responsible for novel types of phase transitions and universality classes [1–4]. For instance, scale-free networks present a complex critical behavior for the Ising model, percolation, and spreading processes that explicitly depends on the exponent of the power law in the degree distributions [1–3]. On the other hand, the existence of nontrivial spectral properties is crucial for the stability of synchronization processes and O(n) models [4].

Despite the large amount of interest in the subject, much smaller attention has been devoted to critical phenomena on complex networks embedded in a metric space [5–9], though some important problems related to navigability, efficiency, and search optimization in spatial networks have already been discussed in the literature [10–13]. In fact, spatial embedding is a very relevant aspect of infrastructure and technological networks, including airport networks, the Internet, and power-grid networks. Moreover, a pivotal role in shaping the topology of social networks is played by hidden metric structures in some underlying abstract space, such as that of the social distance between individuals [8,9].

The aim of this Letter is to investigate the role of spatial embedding in relation with the critical behavior of phase transitions in complex networks. It is well known that in regular lattices, space dimensionality governs the critical behavior of equilibrium and nonequilibrium systems. In particular, below the upper critical dimension, critical fluctuations that are not captured by the mean-field approach set in. Similarly, for complex networks embedded in a low dimensional space we can expect that, as the link probability becomes short ranged, the effect of the underlying space might change the critical behavior leading to a breakdown of the validity of (heterogeneous) mean-field arguments. This should be relevant to understand real phenomena in spatial networks, such as the spreading of viruses [6], the emergence of congested phases in the packet-based traffic on technological networks [14], and cascading failure phenomena in power-grid networks [15].

As a prototypical example of the complex behavior induced by spatial embedding, in this Letter we consider the Ising model on annealed scale-free networks. On a scale-free network with a degree distribution $P(k) \sim k^{-\gamma_{\mathrm{SF}}}$, the critical temperature of the Ising model diverges for $\gamma_{\mathrm{SF}} < 3$. The critical exponents, computed by means of the annealed network approximation [16] or by assuming a quenched randomness [17,18], deviate from the mean-field ones as long as $\gamma_{\mathrm{SF}} < 5$, with the exception of γ, γ' describing the divergence of the magnetic susceptibility χ close to the critical temperature T_c ($\chi \sim |T - T_c|^{-\gamma,\gamma'}$). In fact, γ, γ' always remain fixed to their mean-field value $\gamma = \gamma' = 1$. For these reasons we refer to the critical behavior of random scale-free networks as the *heterogeneous mean-field* solution. We derive here a *Ginsburg criterion* [19] for spatial complex networks determining the condition under which critical fluctuations become larger than the ones predicted within a mean-field approach. In particular, we will show that the critical behavior is always mean field, whenever the matrix $\mathbf{p} = \{p_{ij}\}_{i,j=1,\dots,N}$, fixing the probabilities of existence of each link (i, j) has a finite spectral gap Δ between the maximal eigenvalue Λ and the second maximal one λ_2. On the contrary, when the spectral gap $\Delta \to 0$ in the thermodynamic limit, the critical behavior depends on the behavior of the tail of the spectrum of $\mathbf{p}$. We will demonstrate by theoretical and numerical results that the behavior of such a tail is well captured by an exponent δ_S, related to the effective dimension d_{eff} of the network through the relation $\delta_S = (d_{\mathrm{eff}} - 2)/2$. We find that for

$\delta_S < 1$ the critical fluctuations become dominant, and close enough to the critical temperature the mean-field theory is not sufficient to correctly characterize the critical exponents, possibly calling for renormalization group calculations.

Networks with spatial embedding.—We consider networks of N nodes embedded in a d-dimensional Euclidean metric space, each node $i = 1, \ldots, N$ having position $\vec{r}_i$. The minimal hypothesis [20] that can be made on random networks with heterogeneous degrees and spatial embedding is that links (i, j) are drawn with probability p_{ij} given by

$$p_{ij} = \frac{\theta_i \theta_j J(\vec{r}_i, \vec{r}_j)}{1 + \theta_i \theta_j J(\vec{r}_i, \vec{r}_j)} \simeq \theta_i \theta_j J(\vec{r}_i, \vec{r}_j), \tag{1}$$

where we assumed that $[\max_i(\theta)]^2[\max_{\vec{r},\vec{r}'} J(\vec{r}, \vec{r}')] \ll 1$ and that the matrix $J(\vec{r}_i, \vec{r}_j)$ only depends on the distance between the nodes, i.e., $J(\vec{r}_i, \vec{r}_j) = J(|\vec{r}_i - \vec{r}_j|)$. In this ensemble the degree k_i of a node i is a Poisson random variable with expected degree $\bar{k}_i$ fixed by means of the hidden variables θ_i and given by the relation $\bar{k}_i = \sum_j p_{ij}$. Therefore, given a set of expected degrees $\{\bar{k}_i\}$, we can evaluate the $\{\theta_i\}$ variables by solving the equations $\bar{k}_i = \sum_j p_{ij}$. Networks with homogeneous degrees are generated by fixing $\theta_i = \theta\ \forall\ i$, which corresponds to the Manna-Sen model of spatial networks [7]. Another special choice is that of space-independent couplings $J_{ij} = J\ \forall\ i, j = 1, \ldots, N$, which gives $\theta_i = \bar{k}_i/\sqrt{J\langle\bar{k}\rangle N}$, where $\langle\bar{k}\rangle = \sum_i \bar{k}_i/N$. In this case, our formalism easily recovers known results for both the percolation threshold and the critical temperature of the Ising model on complex networks without spatial embedding.

Ising model on annealed spatial networks and the Ginsburg criterion.—We consider a system of binary spin variables $s_i = \pm 1$, for $i = 1, \ldots, N$, defined on the nodes of a given annealed network with spatial embedding and link probability given by the matrix $\mathbf{p}$. The partition function [1,16] for this problem is given by

$$Z = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})}, \tag{2}$$

with

$$H(s_i) = -\frac{1}{2}\sum_{i \neq j} s_i \theta_i J_{ij} \theta_j s_j - \sum_i H_i s_i. \tag{3}$$

In order to derive a Ginsburg criterion for this statistical mechanics problem, we generalized the classical approach by means of stationary phase approximation [19]. Considering only the first-order terms in the expansion leads to mean-field results. Thus the validity of the mean-field solution can be checked by evaluating the higher order corrections at the critical point. Critical fluctuations that are neglected by the mean-field set in when the second order corrections diverge, dominating the behavior of the susceptibility at criticality. In the stationary phase approximation, the magnetization of the system is given by the m_i^0's satisfying the self-consistent equations:

$$m_i^0 = \tanh\left[\beta\left(H_i + \sum_j \theta_i J_{ij} \theta_j m_j^0\right)\right]. \tag{4}$$

At the second order of the stationary phase approximation [19], performing a Legendre transformation we can evaluate the free energy $\Gamma(\{m_i\})$ as

$$\begin{aligned}\Gamma(\{m_i\}) = &-\frac{1}{2}\sum_{ij} m_i \theta_i J_{ij} \theta_j m_j + \frac{1}{2\beta}\sum_i [(1-m_i)\ln(1-m_i) \\ &+ (1+m_i)\ln(1+m_i)] \\ &+ \frac{1}{2z\beta}\ln\det[\delta_{ij} - \beta J_{ij}\theta_i\theta_j(1-m_j^2)],\end{aligned} \tag{5}$$

where the external field $H_i = \partial\Gamma(\{m_i\})/\partial m_i$ and we have introduced the parameter z in order to keep track of the different orders in the expansion. The susceptibility matrix is defined as $\chi_{i,j}^{-1} = \frac{\partial^2 \Gamma}{\partial m_i \partial m_j}$. We compute it in the paramagnetic phase, where $m_i = 0$, and then we perform the projection along the eigenvector u_i^λ associated with the eigenvalue λ of the connectivity matrix, obtaining

$$\chi_\lambda^{-1} = -\lambda + \frac{1}{\beta} + \frac{1}{z}\sum_{i,\ell} p_{i\ell}[\mathbf{1} - \beta\mathbf{p}]^{-1}{}_{\ell i}(u_i^\lambda)^2, \tag{6}$$

where $\mathbf{1}$ is the identity matrix. The instability of the paramagnetic phase is now determined in terms of the largest eigenvalue Λ of the matrix p_{ij} through the condition $\chi_\Lambda^{-1}(T_c) = 0$. We express the susceptibility in terms of the spectral density $\rho(\lambda)$ of the matrix $\mathbf{p}$ as

$$\chi_\Lambda^{-1}(T) = -\Lambda + T + \frac{1}{z}\int d\lambda \rho(\lambda) f(\lambda) \frac{\lambda}{1 - \frac{\lambda}{T}}, \tag{7}$$

where $f(\lambda) = N\sum_i (u_i^\lambda u_i^\Lambda)^2$. To leading order in $1/z$ the critical temperature T_c is given by

$$T_c = \Lambda - \frac{1}{z}\int d\lambda \rho(\lambda) f(\lambda) \frac{\lambda}{1 - \frac{\lambda}{\Lambda}}. \tag{8}$$

Using (8), we can express the susceptibility, Eq. (7), for $T \to T_c$ as

$$\frac{\chi^{-1}}{T - T_c} = \left[1 - \frac{1}{z}\int d\lambda \frac{\rho(\lambda) f(\lambda)(\lambda)^2}{(T - \lambda)(T_c - \lambda)}\right]. \tag{9}$$

We assume now that the spectrum $\rho(\lambda)$ has a spectral edge λ_c equal to the average value of the second largest eigenvalue λ_2 of $\mathbf{p}$, i.e., $\lambda_c = \langle\lambda_2\rangle$ such that the spectrum for $\lambda < \lambda_c$ is self-averaging. For $\lambda < \lambda_c$, close to the upper edge, we assume the scaling behavior

$$\rho(\lambda) \simeq (\lambda_c - \lambda)^{\delta_S}, \tag{10}$$

which we can use to perform the integral in (9). Moreover, we define the spectral gap Δ_N of a network of size N as the difference between the maximal eigenvalue Λ and the spectral edge, i.e., $\Delta_N = \Lambda - \lambda_c$. Performing a straightforward calculation under the assumption that the gap Δ_N is self-averaging in the thermodynamic limit, i.e., $\lim_{N\to\infty}\Delta_N \to \Delta$, we distinguish two possible behaviors. If $\Delta > 0$, close to the critical temperature $T \to T_c$, we have

$$\chi^{-1} = (T - T_c)[1 - (\Delta^{\delta_S - 1}\mathcal{C}_2 - \mathcal{C}_1)/z], \tag{11}$$

where $\mathcal{C}_{1,2}$ are constants. In this case the critical fluctuations are always mean field. On the other hand, if $\Delta = 0$, we have

$$\chi^{-1} = (T - T_c)[1 - (T - T_c)^{\delta_S - 1}\mathcal{C}_3/z + \mathcal{C}_1/z], \tag{12}$$

with constants $\mathcal{C}_{1,3}$. In this case the critical behavior depends on the particular value of δ_S. For $\delta_S \geq 1$ the corrections of order $1/z$ to χ^{-1} do not modify the critical behavior of the susceptibility. On the contrary, for $\delta_S < 1$, the corrections of order $1/z$ diverge close to the phase transition, the fluctuations dominate the critical behavior, and the mean-field approach cannot be applied. As a first check we look at the case of homogeneous degree distributions. We consider a d-dimensional lattice of linear size L, homogeneous hidden variables $\theta_i = \theta \;\forall\; i$, and coupling matrices

$$J(\vec{r}_i, \vec{r}_j) = \exp(|\vec{r}_i - \vec{r}_j|/d_0), \tag{13}$$

depending on the typical distance d_0. In this case we always get $\lim_{N\to\infty}\Delta_N = 0$ and $\delta_S = (d-2)/2$, recovering the classical result of the Ginsburg criterion which states that the critical dimension for the Ising model is $d = 4$.

Application to complex spatial networks.—We now turn to the case of linking matrices **p** describing annealed scale-free networks embedded in a d-dimensional space with finite critical temperature T_c. For the sake of concreteness, we consider a regular $d = 2$ lattice of side L, we assign to each of the $N = L^2$ nodes an expected degree $\bar{k}$ according to a power-law distribution $p(\bar{k}) \sim \bar{k}^{-\gamma_{\rm SF}}$, and we consider exponentially decaying couplings as in Eq. (13). The values of the parameters $\{\theta_i\}$ are given by the solution of the set of equations $\bar{k}_i = \sum_j p_{ij}$ with $p_{ij} = \theta_i\theta_j J_{ij}$.

The role played by spatial embedding in the critical behavior of these networks is well characterized by the spectrum $\rho(\lambda)$ of the corresponding matrix **p**. For small d_0, where we expect nontrivial effects of space, the behavior of the spectrum close to λ_c follows Eq. (10). In Fig. 1 we report the cumulative distribution (rank plot) of the eigenvalues of **p** for $d_0 = 1$ and different values of $\gamma_{\rm SF}$. We observe that the spectral density below the spectral edge is self-averaging and the exponent δ_S is a decreasing function of $\gamma_{\rm SF}$ (at constant d_0) assuming values above and below $\delta_S^\star = 1$. However, the maximal eigenvalue Λ and the spectral gap Δ are not, in general, self-averaging, being subject to strong fluctuations also for large network sizes. This occurs also for the parameter values studied in Fig. 1.

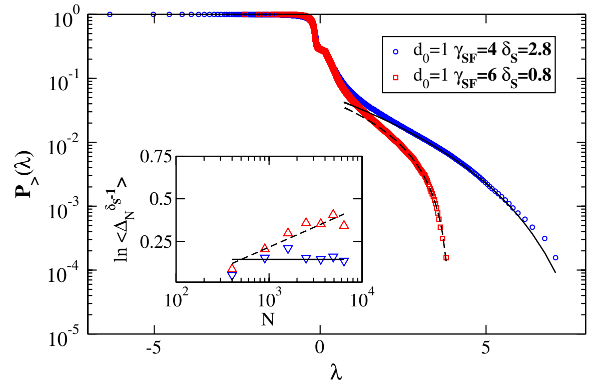

FIG. 1 (color online). Cumulative rank plot of the averaged spectra of 100 matrices **p** for scale-free random networks embedded in dimension $d = 2$, linear size $L = 80$, coupling scale $d_0 = 1$, minimal expected degree $m = 2$, and $\gamma_{\rm SF} = 4, 6$. The behavior for large eigenvalues is well fitted by the expression (10) with different exponent δ_S below and above the critical value $\delta_S^\star = 1$. Inset: Average value $\langle \Delta_N^{\delta_S - 1}\rangle$ for network ensembles with the same parameters as before but with varying system size $N = L^2$. The fit shows that for $\gamma_{\rm SF} = 6$ the quantity $\langle \Delta_N^{\delta_S - 1}\rangle$ increases with the system size, while for $\gamma_{\rm SF} = 4$ it remains constant.

The absence of self-averaging is also observed for networks without spatial embedding, where it is essentially driven by the cutoff fluctuations [21]. While this *anomalous* effect might also be present in spatial networks, it seems that the sample-to-sample fluctuations observed in the spectral gap are mainly due to a new feature of spatial networks, i.e., their local geometry. In fact the non-self-averaging properties also appear for values of the exponent $\gamma_{\rm SF}$ (for example, $\gamma_{\rm SF} = 6$), where the fourth moment of the degree converges and the critical behavior associated with a complex network without spatial embedding is self-averaging [21]. We checked numerically in a number of cases that the spectral gap is non-self-averaging, but the probability $P(\Delta_N^{\delta_S - 1})$ is stable when the value of $\Delta_N^{\delta_S - 1}$ is rescaled with its average value $\langle \Delta^{\delta_S - 1}\rangle$ (see Fig. 2). Therefore, in this case we characterize the average critical behavior of the ensemble by the quantity

$$\Psi = \lim_{N\to\infty}\left\langle \frac{\chi^{-1}}{T - T_c^{\rm eff}} \right\rangle = \lim_{N\to\infty}\left[1 - \frac{\langle \Delta_N^{\delta_S - 1}\rangle \mathcal{C}_2 - \mathcal{C}_1}{z}\right], \tag{14}$$

where $T_c^{\rm eff}$ is the effective critical temperature of a network and depends explicitly on the size N. If Ψ diverges, i.e., $\lim_{N\to\infty}\langle \Delta_N^{\delta_S - 1}\rangle \to \infty$, we expect that the critical fluctuations neglected by the mean-field approach become relevant. In the inset of Fig. 1 we report $\langle \Delta_N^{\delta_S - 1}\rangle$ averaged over 100 realizations of the **p** matrices for the two network ensembles with $d_0 = 1$, $\gamma_{\rm SF} = 4, 6$ as a function of the network size N. The results for $d_0 = 1$, $\gamma_{\rm SF} = 4$ are compatible with a limit $\langle \Delta_N^{\delta_S - 1}\rangle \to \text{const}$ for $N \to \infty$.

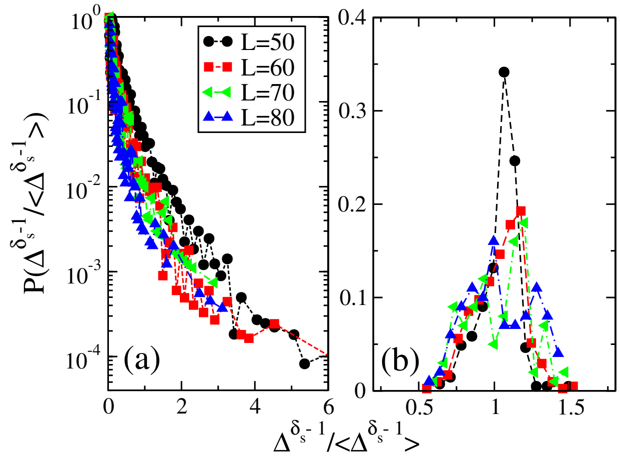

FIG. 2 (color online). The distribution $P(\Delta^{\delta_S - 1})$ is not self-averaging but is a stable distribution when the variable $\Delta^{\delta_S - 1}$ is normalized with its average value. Fixing the value of $d_0 = 1$, we show in (a) the distribution for $\gamma_{SF} = 4$, while in (b) $\gamma_{SF} = 6$. Both figures are obtained from a diagonalization of M matrices of size $N = L^2$ with $L = 50, 60, 70, 80$. In particular, the number of samples M is, respectively, $M = 1000$ for $L = 50$, $M = 400$ for $L = 60$, $M = 100$ for $L = 70$, and $M = 100$ for $L = 80$.

Therefore, in this case, the critical behavior should be well captured by the mean-field behavior. For networks with $d_0 = 1$ and $\gamma_{SF} = 6$, instead, $\langle \Delta_N^{\delta_S - 1} \rangle$ seems to diverge as $N \to \infty$, signaling the presence of critical fluctuations not captured by the mean-field approach.

Conclusions.—In this Letter we have investigated how spatial embedding can affect the critical behavior around a phase transition in systems defined on spatial complex networks. In particular, by means of a detailed study of the Ising model on annealed spatial complex networks, we have shown that relevant critical fluctuation not captured by any (heterogeneous) mean-field theory may set in. Our analysis points out that knowledge of the spectral properties of the link probability matrix $\mathbf{p}$ is crucial for the understanding of the critical behavior of dynamical processes and suggests a classification of the latter based on a generalized Ginsburg criterion. More precisely, when the spectrum presents a finite gap $\Delta > 0$ in the thermodynamic limit, the fluctuations are always mean field. If instead the gap vanishes in the thermodynamic limit, the critical behavior depends on the exponent δ_S describing the scaling of the spectral density close to its upper edge. A fascinating open problem is the relation between the critical behavior of annealed and quenched spatial networks. The solution of this problem might show other new unexpected effects due to fluctuations of the local geometry. Finally, our results open new perspectives for the comprehension of critical phenomena in spatial complex networks, whereas the general formalism presented here could be applied to the study of realistic models of epidemic spreading in transportation networks as well as to the study of the control of fluctuations in technological and power-grid networks.

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