Exact Spectrum of Planar $N = 4$ Supersymmetric Yang-Mills Theory: Konishi Dimension at Any Coupling

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(Received 18 October 2009; published 28 May 2010)

We compute the full dimension of the Konishi operator in planar $N = 4$ super Yang-Mills theory for a wide range of couplings, from weak to strong coupling regime, and predict the subleading terms in its strong coupling asymptotics. For this purpose we solve numerically the integral form of the AdS/CFT Y-system equations for the exact energies of excited states proposed by us and A. Kozak.

DOI: 10.1103/PhysRevLett.104.211601 PACS numbers: 11.25.Tq, 11.15.Tk

Introduction.—Four-dimensional Yang-Mills (YM) theories are at the heart of modern high energy physics, describing all fundamental interactions except gravity. Nevertheless, in spite of considerable efforts during almost 40 years, we still do not have a satisfactory quantitative description of the most interesting YM theories, such as QCD, in the region of intermediate and strong couplings. The low energy quantum dynamics of YM field is mostly known only from computer simulations of lattice YM theories. A few important exact results concerned the topological, protected sectors of $N = 1$, 2 super Yang-Mills (SYM) theory were obtained.

Recently, when the hopes on complete exact 4D solutions, in particular, for the quantities given by nontrivial 4D Feynman series seemed to start waning, $N = 4$ supersymmetric Yang-Mills theory gave us serious hopes for a better understanding of the dynamics of strongly interacting 4D gauge theories. Because of the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence [1], as well as to the quantum integrability discovered on both sides of this duality in the planar limit (when the number of colors $N \to \infty$ with the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ fixed) [2–11], we acquire little by little tools for the study of the most important quantities in $N = 4$ SYM theory, such as the spectrum of dimensions $\Delta(\lambda)$ of local operators as functions of λ—the scale independent coupling constant in this superconformal 4D theory. The weak coupling behavior ($\lambda \to 0$) of $\Delta(\lambda)$ is given by Feynman perturbation theory. The dual string world sheet σ model with the coupling $g = \sqrt{\lambda}/4\pi$ allows us to find the strong coupling asymptotics of various dimensions. Integrability allows us to connect the two regimes. In particular, the asymptotic Bethe ansatz (ABA) of [12] gives us the asymptotic spectrum of single trace operators containing an asymptotically large number of elementary fields.

However, for short operators, such as the Konishi operator $\mathrm{Tr}[D, Z]^2$ [13,14], the calculation of anomalous dimensions is still an interesting and important challenge.

Recently we proposed the Y system for the planar AdS/CFT [15], a set of functional equations defining the anomalous dimensions of *all* operators of planar $N = 4$ SYM theory at any coupling. The integral form of the Y system for excited states in $SL(2)$ sector, including the one corresponding to the Konishi operator, was presented in [16]. The integral equations for the protected vacuum energy were independently obtained in [17,18] by the thermodynamical Bethe ansatz (TBA) technique based on the dynamics of bound states [19–25] (see also [9,26]) of the mirror theory [27,28]. The solutions of the integral Y system are also solutions of the functional Y system [16–18,29]. The combination of functional and integral versions of the Y system appears to be quite efficient to compute numerically the exact spectrum. In this work, we use the functional form of the Y system to derive the large volume (L) [30] asymptotic solution and then, departing from it, we solve the integral form of the Y system iteratively. As a demonstration of the power of our method, we calculate numerically the dimension of the Konishi operator in a wide range of the 't Hooft coupling covering both the weak and strong coupling regimes. The results appear to be quite satisfactory: we manage to compute the dimension of the Konishi operator in the interval of couplings $0 \lesssim \lambda \lesssim 700$ and to confirm, within the precision of our numerics, all the existing data concerning this quantity: the perturbative results [31] up to 4 loops (up to λ^4 terms) [32–35] and the large λ asymptotics $2\lambda^{1/4}$ matches the prediction of [36] for the lowest level of the string spectrum. Fitting our numerical data with $\lambda > 60$ we find (with uncertainty in the last digit)

$$\Delta_K = 2\lambda^{1/4}\left(1.0002 + \frac{0.994}{\lambda^{1/2}} - \frac{1.30}{\lambda} + \frac{3.1}{\lambda^{3/2}} + \ldots\right). \quad (1)$$

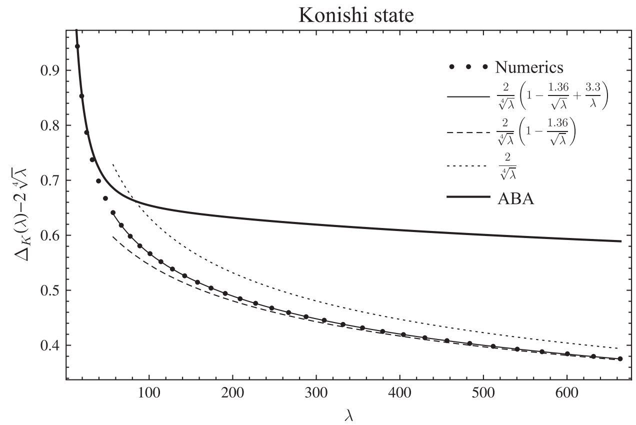

FIG. 1. Plot of $\Delta_K(\lambda) - 2\lambda^{1/4}$ from the numerical data compared with the Bethe ansatz prediction and some fits. The fits in this plot are done assuming the asymptotics $\Delta_K(\lambda) = 2\lambda^{1/4} + 2/\lambda^{1/4} + \ldots$

The leading term reproduces indeed the expected large λ behavior within our numerical precision. It was also argued in [37] that the subleading coefficient ought to be an integer (the next corrections could be transcendental [38]). Indeed, our numerics seems to indicate that $\Delta_K = 2\lambda^{1/4} + 2/\lambda^{1/4} + \ldots$ thus predicting the value of this integer. [39] We also obtained predictions for two further subleading corrections (with a lower precision, of course).

Our results are represented in Fig. 1.

Y-system functional and integral equations for AdS/CFT.—The Y system defining the spectrum of all local operators in planar AdS/CFT correspondence reads [15]

$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s}Y_{a-1,s}} = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+Y_{a+1,s})(1+Y_{a-1,s})}. \tag{2}$$

where the functions $Y_{a,s}(u)$ correspond only to the nodes marked by ○, ⊕, ⊗, △, ● in Fig. 2 (we will use these notations for Y functions in what follows). The one particle energy at infinite length $\epsilon^{(a)}(u) = a + \frac{2ig}{x^{[-a]}} - \frac{2ig}{x^{[+a]}}$ is defined through the Zhukovski map $x(u) + \frac{1}{x(u)} = \frac{u}{g}$ and

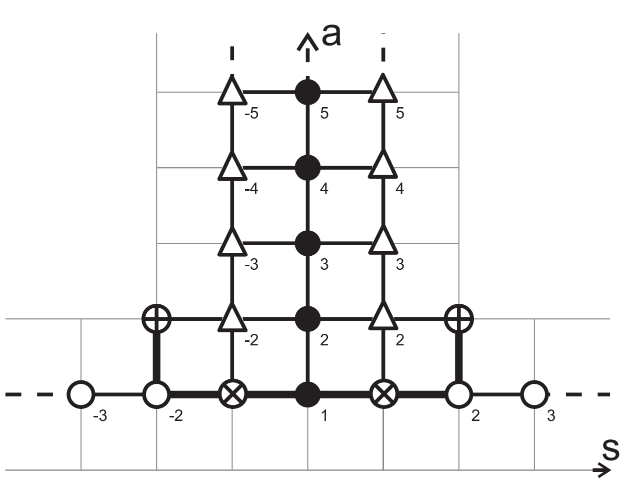

FIG. 2. **T**-shaped domain (**T**-hook) [43]. It defines the interactions between Y's in the AdS/CFT Y-system equations.

$f^{[\pm a]} \equiv f(u \pm ia/2)$, $f^{\pm} \equiv f(u \pm i/2)$ for any function $f(u)$. A solution of Y system with a given set of quantum numbers defines the energy of a state (or dimension of an operator in $N = 4$ SYM theory) through the formula (3) where the Bethe roots u_j are fixed by the exact Bethe ansatz equations $Y_{\bullet_1}(u_j) = -1$. In this Letter we restrict ourselves to the integral form of the Y system for the $SL(2)$-excited states obtained in [16]. Furthermore, we focus ourself for simplicity on the Konishi operator where we have only two roots $u_1 = -u_2$ which we can encode into the "Baxter functions" $R^{(\pm)}(u) = [x(u) - x_1^{\mp}] \times [x(u) - x_2^{\mp}]$ and their complex conjugates $B(u) = \bar{R}(u)$ where $x_{1,2}^{\mp} = x(u_{1,2} \mp i/2)$ with the *physical* choice of branches, such that $|x(u)| > 1$, while for the free variable $x(u)$ we should use the *mirror* kinematics which corresponds to the branches where $\mathrm{Im}(x(u)) > 0$ [28]. Unless it is explicitly said otherwise, we will be always using this latter choice in what follows. With the mirror choice, $x(u)$ has a semi-infinite cut for $u \in (-\infty, -2g) \cup (2g, +\infty)$. The energy of the Konishi state is computed from

$$\Delta_K = 2 + 2\epsilon^{(1)}(u_1) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \partial_u \epsilon^{(a)} \log(1 + Y_{\bullet_a}), \tag{3}$$

where the integral equations defining $Y_{\bullet_a}$ read [16]

$$\begin{aligned}
\log Y_{\otimes} &= K_{m-1} * \log(1 + 1/Y_{\bigcirc_m})/(1 + Y_{\triangle_m}) \\
&\quad + \mathcal{R}^{(0m)} * \log(1 + Y_{\bullet_m}) + \log\frac{-R^{(+)}}{R^{(-)}} \\
\log Y_{\triangle_n} &= \mathcal{M}_{nm} * \log(1 + Y_{\bullet_m}) - K_{n-1} \circlearrowright \log(1 + Y_{\otimes}) \\
&\quad - K_{n-1,m-1} * \log(1 + Y_{\triangle_m}) + \log\frac{R_n^{(+)} B_{n-2}^{(+)}}{R_n^{(-)} B_{n-2}^{(-)}} \\
\log Y_{\bigcirc_n} &= K_{n-1,m-1} * \log(1 + 1/Y_{\bigcirc_m}) + K_{n-1} \circlearrowright \log(1 + Y_{\otimes}) \\
\log Y_{\bullet_n} &= \mathcal{T}_{nm} * \log(1 + Y_{\bullet_m}) + 2\mathcal{R}^{(n0)} \circlearrowright \log(1 + Y_{\otimes}) \\
&\quad + 2\mathcal{N}_{nm} * \log(1 + Y_{\triangle_m}) + i\Phi_n.
\end{aligned} \tag{4}$$

We use here the kernels and sources defined in [16] and presented in the Appendix for completeness. The integrals in convolutions $K * f = \int dv K(u, v) f(v)$ go along the real axis, but slightly below the poles in the terms involving $Y_{\triangle_2}$ (due to the last term in the corresponding integral equation, see [16]). The convolutions $\circlearrowright$ should be understood in the sense of a B cycle (see [16]), e.g., $\mathcal{R}^{(n0)} \circlearrowright \log(1 + Y_{\otimes})$ stands for

$$\int_{-2g}^{2g} dv [\mathcal{R}^{(n0)} \log(1 + Y_{\otimes}) - \mathcal{B}^{(n0)} \log(1 + 1/Y_{\oplus})]$$

where $\frac{1}{Y_{\oplus}}$ is the analytical continuation of $Y_{\otimes}$ across the cut $u \in (-\infty, -2g) \cup (2g, +\infty)$. Summation over the repeated index m is assumed with $m \geq 2$ for $\triangle_{\pm m}$ and $\bigcirc_{\pm m}$, and $m \geq 1$ for $\bullet_m$.

A remarkable feature of all these equations, crucial for the success of our numerics and noticed in [16], is the reality of all Y functions in the integral equations.

Exact Bethe equations.—The Y system integral equations for the functions $Y_{a,s}$ need to be supplemented by the *exact* Bethe equations $Y^{\rm ph}_{\bullet_1}(u_j) = -1$ which reproduce the asymptotic Bethe equations of Beisert and Staudacher in the large L limit [15]. To use this equation, we need to analytically continue the last of Eq. (4) in the free variable u from mirror to physical plane and then evaluate it at $u = u_1$. After some manipulations with contours we find

$$\log Y^{\rm ph}_{\bullet_1}(u_1) = \tilde{\mathcal{T}}_{1m} * \log(1+Y_{\bullet_m}) + \log Z_{\triangle_2}(u_1) + i\Phi_{\rm ph}(u_1) + 2(\mathcal{R}^{(10)}_{\rm ph,mir} \circlearrowright K_{m-1} + K^-_{m-1}) *_{p.v.} \log(Z_{\triangle_m}) + 2\mathcal{R}^{(10)}_{\rm ph,mir} \circlearrowright \log(1+Y_\otimes) - 2\log\frac{u_1 - i/2}{i} - 2\sum_{j=1}^{2}\log\frac{\frac{1}{x_1^+} - x_j^+}{\frac{1}{x_1^-} - x_j^+} \quad (5)$$

where $*_{p.v.}$ stands for principal value integration, $\tilde{\mathcal{T}}_{1m}$ is the dressing phase in the physical kinematics while $i\Phi_{\rm ph}$ is the same as $i\Phi$ but evaluated in the physical region (see Appendix). We used $1/Y_\oplus(u_j \pm i/2) = 0$ [following from (4)] and denoted $Z_{\triangle_m} = (1 + Y_{\triangle_m})(1 - 1/m^2)(u^2 - u_1^2)^{\delta_{m,2}}$ to isolate the poles in $Y_{\triangle_2}$ at $u = u_j$ and to ensure decreasing asymptotics at large u which is of course useful for the numerics. Finally, in contrast to $Y^{\rm ph}_{\bullet_1}(u_1)$, the term $Z_{\triangle_2}(u_1)$ is evaluated in mirror kinematics.

Numerics and its interpretation.—We solve the integral equations (4) iteratively. As the first step of the iterations we use the large L, ABA solutions of the Y system [15]. At each step of iterations we also update the position of the Bethe roots by solving the exact Bethe equation (5). It is important to note that the right-hand side of (5) happens to be purely imaginary within our precision.

We should also truncate the infinite set of Y functions. We explicitly iterate the first 25 $Y_{\triangle_n}$'s and 25 $Y_{\bigcirc_n}$'s at each step. For $Y_{\bullet_n}$ we can truncate the sums much earlier: the first 5 of them are large enough for our precision. Finally, the integrals along the real axis are computed along the stretch $(-X, X)$ with X being a large cutoff. With these approximations, our absolute precision for the energy is around ± 0.001.

We solved the integral equations for a wide range of couplings $0 \lesssim \lambda \lesssim 700 \gg 1$ stretching from the perturbative region up to this value, which is already a deep strong coupling regime [40]. We found no sign of any singularity and the curve looks perfectly smooth. By this reason we believe that any new singularities, such as those related to the Lüscher μ terms, were unlikely even for higher λ (though it would be easy to incorporate them into our code). This seems to be the case perturbatively [41] and our numerics seem to indicate that the integral form of Y system we are solving describes exactly the full spectrum of planar $N = 4$ SYM theory in $SL(2)$ sector.

With a precision of 0.001 we can approximate the Konishi dimension in the range we considered by $\sqrt[4]{g^2+1}\,\frac{252.93h^4+384.74h^3+674.13h^2+128.17h+4}{35.67h^4+51.43h^3+99.71h^2+29.29h+1}$ where $h = g^2/\sqrt{g^2+1}$ and $\lambda = 16\pi^2 g^2$ (this function is just a shorthand for a table of data).

Conclusions.—We presented here a numerical method for solving the Y system for the Konishi dimension in planar $N = 4$ SYM theory. It opens the way to the systematic study of the spectrum of many interesting states at any values of the coupling. We also hope to simplify the Y system in the future using the underlying Hirota integrable dynamics and reducing it to a finite system of integral equations.

The work of N. G. was partly supported by the German Science Foundation (DFG) under the Collaborative Research Center (SFB) 676 and RFFI project Grant No. 06-02-16786. The work of V. K. was partly supported by the ANR Grants INT-AdS/CFT (No. ANR36ADSCSTZ) and GranMA (No. BLAN-08-1-313695) and the Grant RFFI No. 08-02-00287. We thank G. Arutyunov, S. Frolov, R. Hernandez, R. Janik, L. Lipatov, T. Lukowski, E. Lopez, A. Rej, R. Roiban, R. Suzuki, M. Staudacher, A. Tseytlin, and K. Zarembo for interesting discussions. N. G. and P. V. would like to thank CFP for hospitality and computational facilities.

Appendix.—We use: $\mathcal{P}^{(n)}(v) \equiv -\frac{1}{2\pi}\frac{d}{dv}\log\frac{x_v^{[+n]}}{x_v^{[-n]}}$, $K_n \equiv \frac{2n/\pi}{n^2+4u^2}$ and

$$\mathcal{R}^{(nm)}(u,v) \equiv \frac{\partial_v}{2\pi i}\log\frac{x_u^{[+n]} - x_v^{[-m]}}{x_u^{[-n]} - x_v^{[+m]}} - \frac{1}{2i}\mathcal{P}^{(m)}(v),$$

$$\mathcal{B}^{(nm)}(u,v) \equiv \frac{\partial_v}{2\pi i}\log\frac{1/x_u^{[+n]} - x_v^{[-m]}}{1/x_u^{[-n]} - x_v^{[+m]}} - \frac{1}{2i}\mathcal{P}^{(m)}(v),$$

$$\mathcal{M}_{nm} \equiv K_{n-1} \circlearrowright \mathcal{R}^{(0m)} + K^{\neq}_{n-1,m-1},$$

$$\mathcal{N}_{nm} \equiv \mathcal{R}^{(n0)} \circlearrowright K_{m-1} + K^{\neq}_{n-1,m-1},$$

$$K_{nm} \equiv \mathcal{F}^u_n \circ \mathcal{F}^v_m \circ K_2(u-v),$$

$$K^{\neq}_{nm} \equiv \mathcal{F}^u_n \circ \mathcal{F}^v_m \circ K_1(u-v),$$

where the fusion operation $\mathcal{F}^u_n \circ A \equiv \sum_{k=-(n-1/2)}^{n-1/2} A(u + ik)$. Finally, we also use a nice integral representation [16] of the kernel $\mathcal{T}_{n,m}$ given by

$$\mathcal{T}_{n,m}(u,v) = -K_{n,m}(u-v) - \frac{in}{2}\mathcal{P}^{(m)}(v), -2\sum_{a=1}^{\infty}\int\left[\mathcal{B}^{(10)}_{n1}\left(u, w + i\frac{a}{2}\right)\mathcal{B}^{(01)}_{1m}\left(w - i\frac{a}{2}, v\right) + \text{c.c.}\right]dw,$$

where $\mathcal{B}^{(10)}_{nm} = \mathcal{F}^u_n \circ \mathcal{F}^v_m \circ \mathcal{B}^{(10)}$. For the exact Bethe equations we should use this kernel in the mixed representation,

$$\tilde{\mathcal{T}}_{1m} = -\sum_{a=1}^{\infty} 2\mathcal{B}^{(10)}_{\rm ph,mir}\left(u, w + i\frac{a}{2}\right) * \mathcal{B}^{(0m)}_{\rm mir,mir}\left(w - i\frac{a}{2}, v\right) - \sum_{a=1}^{\infty} 2\mathcal{B}^{(10)}_{\rm ph,mir}\left(u, w - i\frac{a}{2} - i0\right) * \mathcal{B}^{(0m)}_{\rm mir,mir}\left(w + i\frac{a}{2}, v\right) - K_{1m}.$$

Finally the source term $\Phi_n(u) = \mathcal{F}_n^u \circ \Phi(u)$, with

$$\Phi(u) = \frac{1}{i}\log\left[\left(\frac{x^-}{x^+}\right)^{L+M} S^2 \frac{B^{(+)+}R^{(-)-}}{B^{(-)-}R^{(+)+}}\right] \quad \text{(A1)}$$

where the Beisert-Eden-Staudacher [12] dressing phase $S(u) = \prod_{j=1}^{2} \sigma(x^{[\pm]}, x_j^{\pm})$ should be taken in the mixed, mirror-physical representation in the integral equations and in the physical-physical representation for $\Phi_{\rm ph}$ appearing in the exact Bethe equations. We use the mirror-physical integral representation [16]

$$\log S = \log\frac{B^{(-)+}}{B^{(+)+}} + \left(\mathcal{B}^{(10)}(u, w + i0) * \mathcal{G} * \log\frac{R^{(+)}(u - i0)}{R^{(-)}(u - i0)} + \text{c.c.}\right) \quad \text{(A2)}$$

where $\mathcal{G}(u) \equiv \frac{\partial_u}{2\pi i}\log\frac{\Gamma(1-iu)}{\Gamma(1+iu)}$ while for the physical-physical representation we can use the integral representation [42]. Finally $\mathcal{K}_{\rm ph,mr}(u, v)$ represents a kernel where we use the physical (mirror) branches for u (v). For Konishi $L = 2$ and it has $M = 2$ derivatives.

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[2] J. A. Minahan and K. Zarembo, J. High Energy Phys. 03 (2003) 013.
[3] I. Bena, J. Polchinski, and R. Roiban, Phys. Rev. D **69**, 046002 (2004).
[4] N. Beisert, C. Kristjansen, and M. Staudacher, Nucl. Phys. B **664**, 131 (2003).
[5] V. A. Kazakov, A. Marshakov, J. A. Minahan, and K. Zarembo, J. High Energy Phys. 05 (2004) 024.
[6] N. Beisert, V. A. Kazakov, K. Sakai, and K. Zarembo, Commun. Math. Phys. **263**, 659 (2006).
[7] R. A. Janik, Phys. Rev. D **73**, 086006 (2006).
[8] M. Staudacher, J. High Energy Phys. 05 (2005) 054.
[9] N. Beisert, Adv. Theor. Math. Phys.. **12**, 945 (2008).
[10] G. Arutyunov, S. Frolov, and M. Staudacher, J. High Energy Phys. 10 (2004) 016.
[11] N. Beisert, R. Hernandez, and E. Lopez, J. High Energy Phys. 11 (2006) 070.
[12] N. Beisert, B. Eden, and M. Staudacher, J. Stat. Mech. (2007) P021.
[13] M. Bianchi, S. Kovacs, G. Rossi, and Y. S. Stanev, J. High Energy Phys. 05 (2001) 042; B. Eden, C. Jarczak, E. Sokatchev, and Y. S. Stanev, Nucl. Phys. B **722**, 119 (2005).
[14] Here Z is one of the complex scalars and D is a covariant derivative in a light cone direction.
[15] N. Gromov, V. Kazakov, and P. Vieira, Phys. Rev. Lett. **103**, 131601 (2009).
[16] N. Gromov, V. Kazakov, A. Kozak, and P. Vieira, arXiv:0902.4458.
[17] D. Bombardelli, D. Fioravanti, and R. Tateo, J. Phys. A **42**, 375401 (2009).
[18] G. Arutyunov and S. Frolov, J. High Energy Phys. 05 (2009) 068.
[19] Al. B. Zamolodchikov, Phys. Lett. B **253**, 391 (1991).
[20] N. Dorey, J. Phys. A **39**, 13 119 (2006).
[21] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models* (Cambridge University Press, Cambridge, England, 1999).
[22] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. Korepin, *The One-Dimensional Hubbard Model* (Cambridge University Press, Cambridge, England, 2005).
[23] V. Bazhanov, S. Lukyanov, and A. Zamolodchikov, Nucl. Phys. B **489**, 487 (1997).
[24] P. Dorey and R. Tateo, Nucl. Phys. B **482**, 639 (1996).
[25] D. Fioravanti, A. Mariottini, E. Quattrini, and F. Ravanini, Phys. Lett. B **390**, 243 (1997).
[26] G. Arutyunov and S. Frolov, J. High Energy Phys. 03 (2009) 152.
[27] J. Ambjorn, R. A. Janik, and C. Kristjansen, Nucl. Phys. B **736**, 288 (2006).
[28] G. Arutyunov and S. Frolov, J. High Energy Phys. 12 (2007) 024.
[29] S. Frolov and R. Suzuki, arXiv:0906.0499.
[30] L is the number of Z fields in an operator in the $SL(2)$ sector.
[31] Meaningful until the convergency radius $g < g_c = 1/4$. At weak coupling, an interesting numerical prediction could be the order g^{10}. However, our numerical error ± 0.001 is larger than $(g_c)^{10}$ and therefore does not allow for such predictions.
[32] Z. Bajnok and R. A. Janik, Nucl. Phys. B **807**, 625 (2009).
[33] Z. Bajnok, R. A. Janik and T. Lukowski arXiv:0811.4448.
[34] F. Fiamberti, A. Santambrogio, C. Sieg, and D. Zanon, Nucl. Phys. B **805**, 231 (2008); V. N. Velizhanin, Phys. Lett. B **676**, 112 (2009).
[35] N. Beisert, T. McLoughlin, and R. Roiban, Phys. Rev. D **76**, 046002 (2007).
[36] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. B **636**, 99 (2002).
[37] R. Roiban, A. Tseytlin, J. High Energy Phys. 11 (2009) 013, (see A. Tseytlin's talk at Shifmania 2009).
[38] A. Tirziu and A. A. Tseytlin, Phys. Rev. D **78**, 066002 (2008).
[39] Assuming the leading coefficient to be $2\lambda^{1/4}$ one gets 1.001 instead of 0.994 for the subleading term, even closer to 1—the value predicted here.
[40] The expected world-sheet perturbation theory expansion at strong coupling is $\lambda^{1/4}(a_1 + a_2/\sqrt{\lambda} + \ldots)$. The first two coefficients a_i—are of order 1 and we assume that all $a_i \sim 1$. In our numerics $1/\sqrt{\lambda} \sim 0.04$. The appropriate extrapolation procedure can easily increase the precision.
[41] Z. Bajnok, A. Hegedus, R. Janik, T. Lukowski, Nucl. Phys. B **827**, 426 (2010).
[42] N. Dorey, D. M. Hofman, and J. M. Maldacena, Phys. Rev. D **76**, 025011 (2007).
[43] V. Kazakov, A. Sorin, and A. Zabrodin, Nucl. Phys. B **790**, 345 (2008).