Defining a New Class of Turbulent Flows

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We apply a method based on the theory of Markov processes to fractal-generated turbulence and obtain joint probabilities of velocity increments at several scales. From experimental data we extract a Fokker-Planck equation which describes the interscale dynamics of the turbulence. In stark contrast to all documented boundary-free turbulent flows, the multiscale statistics of velocity increments, the coefficients of the Fokker-Planck equation, and dissipation-range intermittency are all independent of R_λ (the characteristic ratio of inertial to viscous forces in the fluid). These properties define a qualitatively new class of turbulence.

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Introduction.—Recent experimental observations of fluid flow turbulence generated by fractal grids like the one shown in Fig. 1 have revealed some remarkable properties. Turbulence generated by space-filling fractal square grids is homogeneous and isotropic far enough downstream where it freely decays exponentially and not as a power law as is the case in all previously well-documented boundary-free turbulent flows (regular grid turbulence, wakes, jets, etc.) [1,2]. As predicted by a theoretical study [3], this exponentially decaying turbulence is locked into a single length scale, meaning that the inner Taylor microscale λ and the outer integral length scale L are both proportional to it. As a result, the ratio L/λ stays constant during decay, although the Reynolds number changes [4]. This implies an independence of the ratio of outer to inner length scales L/λ on Reynolds number which means that fractal-generated homogeneous isotropic turbulence is fundamentally incompatible with the usual Richardson-Kolmogorov cascade picture of small-scale turbulence dynamics where, as the Reynolds number increases, the range of scales needed for the turbulence energy to cascade down to scales small enough for dissipation to occur, also increases. In other words, a wider range of length scales is needed for the Richardson-Kolmogorov cascade to cause turbulence to dissipate at higher Reynolds numbers. Indeed, L/λ is proportional to, and therefore increases with, the Taylor length-based Reynolds number R_λ in all boundary-free turbulent flows [1,5] which are not fractal generated.

Standard statistical analysis of small-sale turbulence is based on two-point correlations and their dependence on the distance r between the two points. A central quantity is the longitudinal velocity increment $\xi(r)$,

$$\xi(r) = u(x + r) - u(x), \tag{1}$$

where u denotes the fluctuating velocity component in the direction defined by two points x and $x + r$. As mentioned above and shown in [3,4], fractal-generated turbulence can be such that $\langle \xi(r)^2 \rangle = u_{\rm rms}^2 f(r/l)$ where the brackets $\langle \ldots \rangle$ denote an averaging operation, $u_{\rm rms}$ is the rms of $u(x)$, f is a dimensionless function, and l is a single length scale determined by the fractal grid, independent of R_λ and such that $L \sim l$ and $\lambda \sim l$. This self-preserving form of the second-order structure function is qualitatively very different from the basic Kolmogorov scaling $\langle \xi(r)^2 \rangle = u_{\rm rms}^2 f(r/L, r/\eta) = u_{\rm rms}^2 g(r/L, L/\eta)$ where f and g are dimensionless functions, η is the Kolmogorov microscale, and $L/\eta \sim R_\lambda^{3/2}$ (equivalently $L/\lambda \sim R_\lambda$). This Kolmogorov scaling involves two different length scales, one outer (L) and one inner (η or λ), and is based on the Richardson-Kolmogorov phenomenology which also requires that $g(r/L) \sim (r/L)^{2/3}$ in the intermediate asymptotic range $\eta \ll r \ll L$ as $L/\eta \to \infty$, so that $\langle \xi(r)^2 \rangle \sim (\epsilon r)^{2/3}$ in that range with $\epsilon \sim u_{\rm rms}^3/L$.

These exceptional properties of fractal-generated turbulence, in particular, the apparent absence of a conventional Richardson-Kolmogorov cascade, call for a deeper analysis of its multiscale structure. The study of all structure functions $\langle \xi(r)^n \rangle$ is equivalent to the study of the proba-

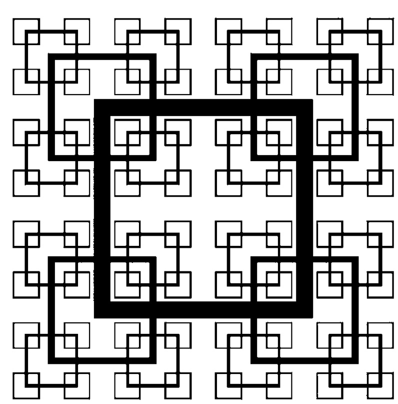

FIG. 1. Fractal square grid.

bility density functions (PDF) $p[\xi(r)]$. Such a study can reveal whether it is in fact $p[\xi(r)]$ which is self-preserving and independent of R_λ or whether it has a Kolmogorov two-scale (inner and outer) dependence and therefore depends on R_λ. However, as there are infinite different possible interscale processes to generate the same one scale $p(\xi)$ statistics, a full multiscale characterization is needed in terms of the multiscale PDF $p(\xi_0, \xi_1, \ldots, \xi_M)$ where $\xi_m \equiv \xi(r_m)$ for $m = 0, 1, \ldots, M$. This multiscale PDF is the joint probability of finding velocity increments on several scales and goes beyond the traditional analysis based on structure functions as $p[\xi(r)]$ can be deduced by integrations. Here we study this multiscale PDF for the purpose of achieving a most general differentiation between fractal-generated turbulence and other boundary-free turbulent flows.

The analysis we use gives access to the multiscale PDF $p(\xi_0, \xi_1, \ldots, \xi_M)$ by characterizing the underlying stochastic interscale process in the form of a Fokker-Planck equation. If the stochastic process for the evolution of the velocity increments from scale to scale ($r_M < r_{M-1} < \ldots < r_0$) has Markov properties, i.e., if

$$p(\xi_M|\xi_{M-1}, \ldots, \xi_0) = p(\xi_M|\xi_{M-1}), \tag{2}$$

the multiscale PDF $p(\xi_M, \ldots, \xi_1, \xi_0)$ can be expressed by a product of conditional PDFs $p(\xi_m|\xi_{m-1})$. The stochastic process for these conditional PDFs can be described by a Kramers-Moyal expansion. If the fourth-order Kramers-Moyal coefficient $D^{(4)}$ is zero, the expansion truncates after the second term (Pawula's theorem) and becomes a Fokker-Planck equation:

$$-\frac{\partial}{\partial r} p(\xi|\xi_0) = -\frac{\partial}{\partial \xi}[D^{(1)}(\xi, r)p(\xi|\xi_0)] + \frac{\partial^2}{\partial \xi^2}[D^{(2)}(\xi, r)p(\xi|\xi_0)], \tag{3}$$

where for simplicity we use the notations $\xi \equiv \xi(r)$ and $\xi_0 \equiv \xi(r_0)$ with $r < r_0$. The drift and diffusion functions $D^{(1)}$ and $D^{(2)}$ can be estimated as Kramers-Moyal coefficients pointwise by

$$D^{(k)}(\xi, r) = \lim_{\Delta r \to 0} \frac{r}{k!\Delta r} \times \int_{-\infty}^{+\infty} (\tilde{\xi} - \xi)^k p[\tilde{\xi}(r - \Delta r)|\xi(r)] d\tilde{\xi}. \tag{4}$$

It has been shown for several different flows [6–10] that (a) the process has Markov properties, (b) $D^{(4)}$ vanishes or is small enough to be neglected, and (c) the experimental (conditional) PDFs of the velocity increments can be reproduced by integration of the Fokker-Planck equation, including intermittency effects.

Experimental results.—We analyze hot-wire measurement data from turbulence generated in a wind tunnel by a fractal square grid shown schematically in Fig. 1. The design of this grid is space filling in the sense that the fractal dimension D_f of the line defined by all the bars without their thickness takes the maximum value 2. The spanwise thickness of these bars determines the blockage ratio independently of the value of D_f, and it is 25% here, which is small compared to regular and active grids. The ratio of the thicknesses of the largest to that of the smallest bars of the grid is $t_r = 17$ [11]. Measurements are taken for two different flow velocities at five different downstream positions in the decay region, where the turbulence is small-scale homogeneous and isotropic [4]. For comparison, we use results from [7,8] for hot-wire measurements of the streamwise velocity component along the center line of a cryogenic free jet, as well as new results from the analysis of hot-wire measurements of the streamwise velocity component in the center of the wake of a cylinder with diameter $D = 2$ cm at downstream-distance $x = 100D$.

For the fractal grid data we confirm the result of [4] that λ is almost independent of downstream position. We find that the stochastic process for the velocity increments has Markov properties for scale separations $\Delta r \equiv r_{m-1} - r_m$ greater than the Einstein-Markov coherence length $l_{\rm EM}$, which is defined as the smallest Δr for which Eq. (2) holds. We estimate $l_{\rm EM}$ with the (Mann-Whitney-)Wilcoxon test, which tests the validity of the equation $p(\xi_2|\xi_1, \xi_0) = p(\xi_2|\xi_1)$ for different values of Δr (cf. [7]). For the fractal grid data, we find a constant ratio of $l_{\rm EM}/\lambda = 0.73 \pm 0.09$, which is comparable to previous results for other turbulent flows, where $l_{\rm EM}/\lambda \approx 0.8$ [9].

We determine the Kramers-Moyal coefficients with two different methods. The first method directly uses definition (4), determining the limit of $\Delta r \to 0$ with a linear fit to the conditional moments on the right-hand side of Eq. (4) in the range $l_{\rm EM} \leq \Delta r \leq 2l_{\rm EM}$, following [7,8]. The drift and diffusion functions at each scale r can then be approximated by linear and second-order functions in ξ, respectively:

$$D^{(1)}(\xi, r) = -d_{11}(r)\xi, \tag{5}$$

$$D^{(2)}(\xi, r) = d_{20}(r) - d_{21}(r)\xi + d_{22}(r)\xi^2. \tag{6}$$

The second method uses numerical optimization to find the optimal coefficients $d_{ij}(r)$ of Eqs. (5) and (6). Here, the Kullback-Leibler entropy is used to minimize the distance between the empirical conditional PDF $p(\xi_m|\xi_{m-1})$, and the conditional PDF obtained by numerical integration of the Fokker-Planck equation (3) [12]. Both the direct and optimization methods lead to consistent results.

The velocity increments $\xi(r)$ are given in units of their standard deviation in the limit $r \to \infty$, σ_∞, which is identical to $\sqrt{2}$ times the standard deviation σ_u of the velocity u [7]. This normalization allows us to compare the Kramers-Moyal coefficients of different flows.

Figure 2(a) shows that the coefficient d_{11} has a similar dependence on r for the fractal grid as for the free jet, and

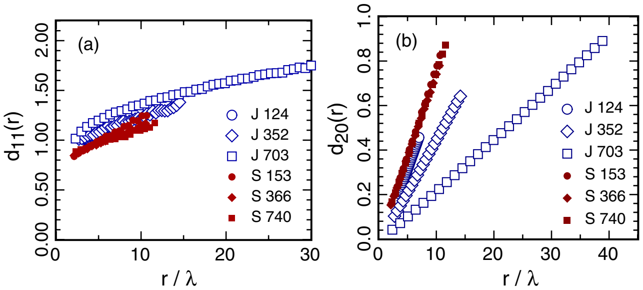

FIG. 2 (color online). Coefficients d_{11} (a) and d_{20} (b) as functions of the scale r for the free jet (J), from [8], and fractal square grid (S). Reynolds numbers R_λ are given in the legend.

does not depend significantly on the Reynolds number for both flows. In contrast, the coefficient d_{20} does depend strongly on R_λ for the free jet [7,8], but not for the fractal grid, as can be seen in Fig. 2(b). The coefficient d_{20} is linear in r and thus can be approximated by

$$d_{20}(r) = d_{20}^{*}\frac{r}{\lambda}. \quad (7)$$

As shown in Fig. 3, d_{20}^{*} follows a power law in R_λ for the free jet [8], as well as for the cylinder wake data where the exponent seems to be smaller. In contrast, the slope d_{20}^{*} is approximately constant for fractal grid turbulence. For both d_{11} and d_{20}, the optimized coefficients differ only slightly from the ones estimated by the classical method.

Renner *et al.* [7,8] also find a strong R_λ dependence of the coefficients d_{21} and d_{22} for free-jet data. For the fractal grid, we find no systematic dependence on R_λ for the optimized values of these coefficients.

As an alternative and independent verification of the Reynolds number independence of the statistical properties of fractal grid turbulence we now investigate the conditional PDFs $p(\xi|\xi_0)$, where $r \ll r_0$. Note that these are the fundamental quantities which contain the information of the stochastic process integrated over a range of scales. Most importantly, the conditional PDFs do not contain the errors and uncertainties which arise in estimating the Kramers-Moyal coefficients.

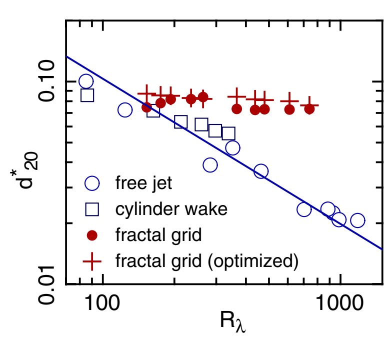

FIG. 3 (color online). Slopes d_{20}^{*} from Eq. (7) as functions of R_λ for the free jet, cylinder wake, and fractal grid. The free-jet data are taken from [8]. The straight line represents a power law $d_{20}^{*} = 2.8Re^{3/8}$ from [8], where $R_\lambda \approx Re^{1/2}$.

Figure 4 shows the conditional PDFs $p(\xi|\xi_0)$, as well as the PDFs $p(\xi)$, which are obtained by integration over ξ_0, for $r = 3l_{\mathrm{EM}}$ and $r_0 = 9l_{\mathrm{EM}}$, for the fractal grid, free jet, and cylinder wake. The PDFs for high and low Reynolds numbers are plotted into the same graph for comparison.

The (conditional) PDFs of the fractal grid data in Fig. 4(a) are practically identical at different Reynolds numbers. The small deviations in the tails of the distributions can be attributed to statistical errors due to the small

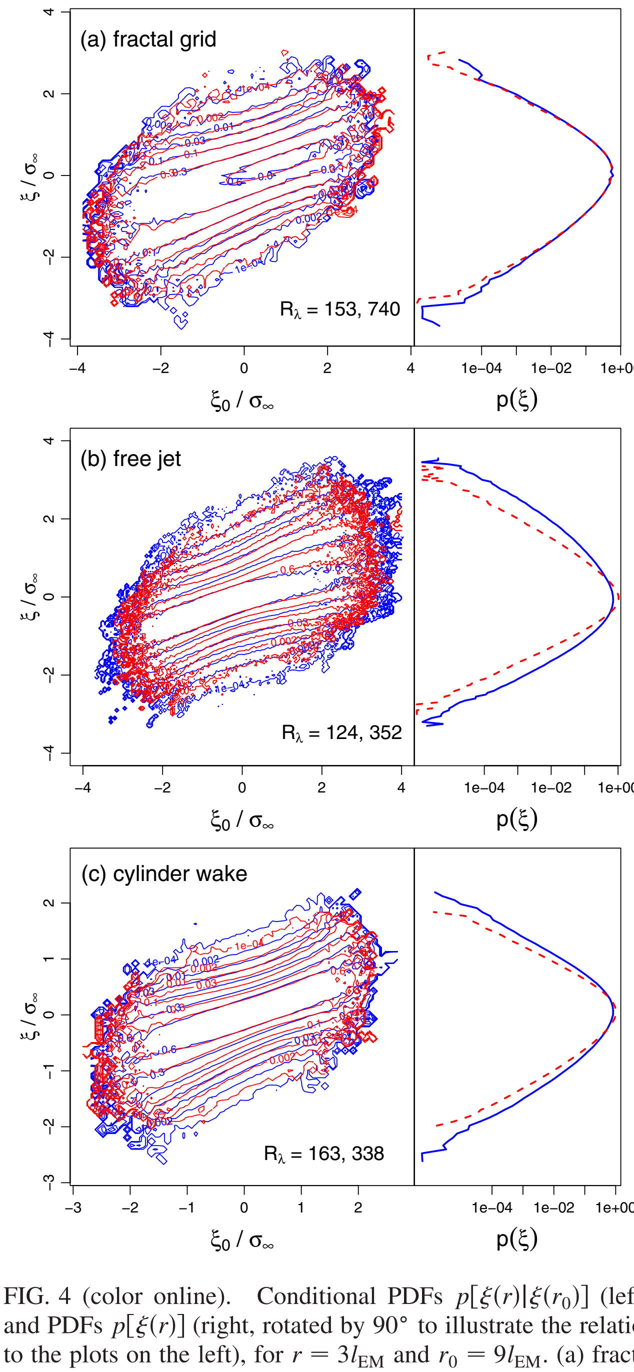

FIG. 4 (color online). Conditional PDFs $p[\xi(r)|\xi(r_0)]$ (left), and PDFs $p[\xi(r)]$ (right, rotated by 90° to illustrate the relation to the plots on the left), for $r = 3l_{\mathrm{EM}}$ and $r_0 = 9l_{\mathrm{EM}}$. (a) fractal grid, $R_\lambda = 153$ (solid), 740 (dashed). (b) free jet, $R_\lambda = 124$ (solid) 352 (dashed). (c) cylinder wake, $R_\lambda = 163$ (solid), 338 (dashed).

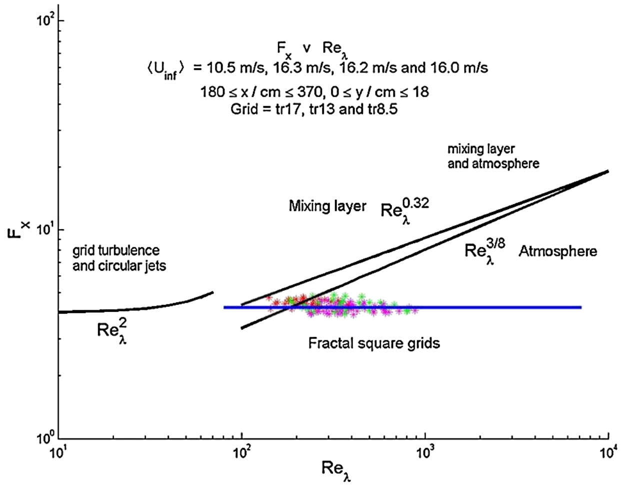

FIG. 5 (color online). Flatness $F_x = \langle u_x^4 \rangle / \langle u_x^2 \rangle^2$ of the velocity derivatives $u_x = \partial u / \partial x$ for different fractal square grids in comparison to other types of turbulence.

number of events in the corresponding bins. In contrast to this, the (conditional) PDFs of the free jet and cylinder wake data in Figs. 4(b) and 4(c), respectively, exhibit large differences for different Reynolds numbers.

The conditional PDFs $p(\xi|\xi_0)$ characterize the interscale dynamics of the small-scale turbulence. Our results strongly suggest that there is a qualitative difference between the interscale dynamics of turbulence generated by space-filling fractal square grids and the interscale dynamics of other boundary-free turbulence such as jet and wake turbulence. The conditional PDFs $p(\xi|\xi_0)$ of our fractal-generated turbulence do not depend on Reynolds number whereas those of other turbulent flows do. This is probably the most fundamental way in which these two different classes of turbulence differ. A particular consequence of the Reynolds number independence of $p(\xi|\xi_0)$ is the Reynolds number independence of $p(\xi)$ which is obtained by integrating $p(\xi|\xi_0)$ over ξ_0. This is confirmed by the right plots of Fig. 4 where it is also shown that $p(\xi)$ is R_λ dependent in jet and wake turbulence.

The method of stochastic analysis applied in this Letter is limited to scales r larger than the Einstein-Markov length l_{EM} ($\approx \lambda$) and therefore so are our conclusions concerning $p(\xi|\xi_0)$. However, if we allow ourselves to extrapolate the Reynolds number independence of $p[\xi(r)]$ to all scales r, then our fractal-generated turbulence is incompatible with Kolmogorov scaling $\langle \xi(r)^n \rangle = u_{\rm rms}^2 g_n(r/L, L/\eta)$ (where $L/\eta \sim R_\lambda^{3/2}$) and must instead obey self-preserving single-length-scale forms $\langle \xi(r)^n \rangle = u_{\rm rms}^n f_n(r/l)$ as previously reported for $n = 2$ [3,4].

The second remarkable consequence of such an extrapolated Reynolds number independence of $p[\xi(r)]$ is the absence of Reynolds number dependent dissipation-range intermittency, unlike all documented flows (see [2,13]). Our data strongly support this conclusion. Figure 5 is a plot of the derivative flatness F_x; it is clear that it does not depend on R_λ, in very stark contrast with all other documented turbulent flows where F_x grows with R_λ [13].

Conclusions.—Homogeneous and isotropic small-scale turbulence generated by a low-blockage space-filling fractal square grid (Fig. 1) is similar to other boundary-free turbulent flows in that the stochastic process for the evolution of the velocity increments from scale to scale has Markov properties for scale separations greater than the Taylor microscale λ. However, this fractal-generated turbulence differs qualitatively from other documented boundary-free turbulent flows [2,7,8,13] in that the resulting drift and diffusion functions (5) and (6) and the multiscale joint probability functions which they determine are all independent of R_λ. The single-scale probability density function of velocity increments is also independent of R_λ. This implies the absence of (inner and outer) Kolmogorov scaling and of R_λ-dependent dissipation-range intermittency. These properties are in stark contrast with all documented turbulent flows [2,7,8,13]. Thus we believe we have found a qualitatively new class of fluid flow turbulence.

These findings have significant implications for the issue of universality and pave the way for hitherto inconceivable studies on the very conditions which allow the Richardson-Kolmogorov cascade to hold or not, over and above the intermittency corrections usually studied.

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[1] H. Tennekes and J. L. Lumley, *A First Course in Turbulence* (MIT Press, Cambridge, MA, 1972).
[2] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[3] W. K. George and H. Wang, Phys. Fluids **21**, 025108 (2009).
[4] R. E. Seoud and J. C. Vassilicos, Phys. Fluids **19**, 105108 (2007).
[5] S. B. Pope, *Turbulent Flows* (Cambridge University Press, Cambridge, 2000).
[6] R. Friedrich and J. Peinke, Phys. Rev. Lett. **78**, 863 (1997).
[7] C. Renner, J. Peinke, and R. Friedrich, J. Fluid Mech. **433**, 383 (2001).
[8] C. Renner, J. Peinke, R. Friedrich, O. Chanal, and B. Chabaud, Phys. Rev. Lett. **89**, 124502 (2002).
[9] S. Lück, C. Renner, J. Peinke, and R. Friedrich, Phys. Lett. A **359**, 335 (2006).
[10] M. Tutkun and L. Mydlarski, New J. Phys. **6**, 49 (2004).
[11] D. Hurst and J. C. Vassilicos, Phys. Fluids **19**, 035103 (2007).
[12] A. P. Nawroth, J. Peinke, D. Kleinhans, and R. Friedrich, Phys. Rev. E **76**, 056102 (2007).
[13] K. R. Sreenivasan and R. A. Antonia, Annu. Rev. Fluid Mech. **29**, 435 (1997).