Consistent Extension of Hořava Gravity

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We propose a natural extension of Hořava's model for quantum gravity, which is free from the notorious pathologies of the original proposal. The new model endows the scalar graviton mode with a regular quadratic action and remains power-counting renormalizable. At low energies, it reduces to a Lorentz-violating scalar-tensor gravity theory. The deviations with respect to general relativity can be made weak by an appropriate choice of parameters.

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Introduction.—Recently, Hořava has proposed a new approach to quantum gravity [1]. The key idea is to abandon local Lorentz invariance as fundamental and to assume instead that it appears at low energies as an approximate symmetry. The breaking of Lorentz invariance is achieved by equipping the space-time with a preferred foliation by three-dimensional spacelike surfaces, which defines the splitting of coordinates into space and time. This allows us to complete the action of general relativity (GR) with higher spatial derivatives of the metric, improving the UV behavior of the graviton propagator and making the theory power-counting renormalizable. Besides, the action remains second order in time derivatives, avoiding the ghosts of covariant theories of higher-derivative gravity [2].

The concrete realization of this idea as developed in [1] unfolds as follows. One considers the 3 + 1 decomposition of the space-time metric in the preferred foliation,

$$ds^2 = (N^2 - N_i N^i)dt^2 - 2N_i dx^i dt - \gamma_{ij} dx^i dx^j,$$

and writes a generic action of the form [3]

$$S = \frac{M_P^2}{2} \int d^3x dt \sqrt{\gamma} N (K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}[\gamma_{ij}]), \quad (1)$$

where M_P is the Planck mass; K_{ij} is the extrinsic curvature tensor $K_{ij} = (\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i)/(2N)$, with trace K; γ is the determinant of the spatial metric γ_{ij}; λ is a dimensionless constant. The potential term $\mathcal{V}[\gamma_{ij}]$ depends on γ_{ij} and its spatial derivatives and is invariant under three-dimensional diffeomorphisms. Explicitly,

$$\mathcal{V} = -\xi R + M_P^{-2}(A_1 \Delta R + A_2 R_{ij}R^{ij} + \ldots) + M_P^{-4}(B_1 \Delta^2 R + B_2 R_{ij}R^{jk}R^i_k + \ldots), \quad (2)$$

where R_{ij}, R are the Ricci tensor and the scalar curvature constructed out of the metric γ_{ij}; $\Delta \equiv \gamma^{ij}\nabla_i\nabla_j$, and ξ, A_n, B_n are constants. The ellipses represent other three-dimensional diffeormorphism-invariant operators of dimension 4 and 6. As discussed in [1], restricting to the operators of dimensions up to 6 is sufficient to make the theory renormalizable by power counting. In what follows we set $\xi = 1$, which can always be achieved by a suitable rescaling of time. Notice that we do not impose the "detailed balance" of [1].

The action (1) reduces to that of GR if $\lambda = 1$ and the terms of dimension 4 and 6 in $\mathcal{V}$ vanish. Otherwise the model explicitly breaks general covariance down to the subgroup of coordinate transformations

$$\mathbf{x} \mapsto \tilde{\mathbf{x}}(t, \mathbf{x}), \qquad t \mapsto \tilde{t}(t), \quad (3)$$

with the standard transformation rules for the metric components. This invariance fixes the kinetic part of the action (1) to be a function of K_{ij}. The symmetry (3) allows us to restrict N to depend only on time. In this way one obtains the "projectable" version of the theory (as opposed to the generic or "nonprojectable" version).

The introduction of terms with higher spatial derivatives in the action leads to different scaling dimensions of space and time in the UV and results in a (naïvely) power-counting renormalizable theory [1]. Indeed, considering the anisotropic scaling transformation

$$\mathbf{x} \mapsto b^{-1}\mathbf{x}, \qquad t \mapsto b^{-3}t, \quad (4a)$$

$$N \mapsto N, \qquad N_i \mapsto b^2 N_i, \qquad \gamma_{ij} \mapsto \gamma_{ij}, \quad (4b)$$

the kinetic part of the action (1) and the operators of dimension 6 in $\mathcal{V}$ are left unchanged; i.e., they are marginal [4]. The rest of the operators in $\mathcal{V}$ are relevant deformations. From standard arguments, the action constructed from such operators is perturbatively renormalizable. At low energies the potential is dominated by the operator of the lowest dimension, R. This leads to the recovery in the infrared of the relativistic scaling dimension -1 for both space and time.

At low energies, the resulting action differs from that of GR only by the presence of the parameter λ. This suggests that the theory might have GR as its low-energy limit, if λ flows to its GR value $\lambda = 1$ in the infrared. However, this argument is not correct. As pointed out in [1], the explicit breaking of general covariance by the preferred foliation introduces a new scalar degree of freedom in addition to the usual helicity-2 polarizations of the graviton. The study of the properties of this extra mode has revealed that it persists down to low energies and exhibits a pathological behavior which invalidates the consistency of the theory based on (1). The pathologies include strong coupling at energies above a very low-energy scale and fast instabilities, and appear both in the nonprojectable [5–7] and projectable [7–9] cases. These problems can be traced back to the anomalous structure of the quadratic action for the new scalar mode around smooth backgrounds, such as Minkowski space-time. The purpose of this Letter is to show that, as suggested in [7], the quadratic Lagrangian may be regular if the action (1) is supplemented by certain terms.

Improved behavior of the extra mode.—Let us focus on the nonprojectable version of the Hořava model and consider the following 3-vector

$$a_i \equiv \partial_i \ln N, \tag{5}$$

having the geometrical meaning of the proper acceleration of the vector field of unit normals to the foliation surfaces [7]. This vector is manifestly covariant under the transformations (3). Thus, the potential (2) can be extended to include terms depending on a_i [10]. Clearly, a_i has dimension 1 with respect to the scaling (4). To ensure power-counting renormalizability one should add to the potential (2) all the operators of dimensions up to 6, i.e., a new piece

$$\delta\mathcal{V}[\gamma_{ij}, a_i] = -\alpha a_i a^i + M_P^{-2}(C_1 a_i \Delta a^i + C_2 (a_i a^i)^2 + C_3 a_i a_j R^{ij} + \ldots) + M_P^{-4}(D_1 a_i \Delta^2 a^i + D_2 (a_i a^i)^3 + D_3 a_i a^i a_j a_k R^{jk} + \ldots). \tag{6}$$

Note that the operators with odd dimensions are forbidden by spatial parity. Similarly, the terms in the action with one time derivative of a_i are excluded by the time-reversal invariance. Finally, terms with two or more time derivatives acting on a_i have dimension larger than 6 and hence are not allowed by power counting.

The addition of terms of the type (6) to the action (1) endows the extra scalar mode with a healthy quadratic action at all energy scales. To show it, let us consider a flat background. The inequivalent terms in the potential that contribute to the quadratic Lagrangian are:

$$\text{(dim2)} \qquad R,\ a_i a^i, \tag{7a}$$

$$\text{(dim4)} \qquad R_{ij}R^{ij},\ R^2,\ R\nabla_i a^i,\ a_i \Delta a^i, \tag{7b}$$

$$\text{(dim6)} \qquad (\nabla_i R_{jk})^2,\ (\nabla_i R)^2,\ \Delta R \nabla_i a^i,\ a_i \Delta^2 a^i. \tag{7c}$$

The presence of a new operator of dimension 2 with respect to the model of [1] will play a key role in the consistency of the theory at low energies. Introducing the scalar perturbations of the metric $N = 1 + \phi$,

$$N_i = \frac{\partial_i}{\sqrt{\Delta}} B, \qquad \gamma_{ij} = \delta_{ij} - 2\left(\delta_{ij} - \frac{\partial_i \partial_j}{\Delta}\right)\psi - 2\frac{\partial_i \partial_j}{\Delta} E,$$

the quadratic Lagrangian becomes

$$\mathcal{L}^{(2)} = \frac{M_P^2}{2}\Big\{-2\dot{\psi}^2 - 2\psi\Delta\psi + 4\phi\Delta\psi + 4\psi\sqrt{\Delta}\dot{B} + 4\psi\ddot{E} - (\lambda-1)(\sqrt{\Delta}B + \dot{E} + 2\dot{\psi})^2 + \alpha(\partial_i\phi)^2 - \frac{f_1}{M_P^2}(\Delta\psi)^2 - \frac{2f_2}{M_P^2}\Delta\phi\Delta\psi - \frac{f_3}{M_P^2}(\Delta\phi)^2 - \frac{g_1}{M_P^4}\psi\Delta^3\psi - \frac{2g_2}{M_P^4}\phi\Delta^3\psi - \frac{g_3}{M_P^4}\phi\Delta^3\phi\Big\}, \tag{8}$$

where the constants α, f_n, g_n are related to the coefficients in front of the operators (7) in the potential. In particular, α is the coefficient in front of the operator $a_i a^i$. Fixing the gauge $B = 0$ and integrating out the nondynamical fields E and ϕ, we obtain

$$\mathcal{L}^{(2)} = \frac{M_P^2}{2}\left\{\frac{2(3\lambda-1)}{\lambda-1}\dot{\psi}^2 + \psi\frac{P[M_P^{-2}\Delta]}{Q[M_P^{-2}\Delta]}\Delta\psi\right\}, \tag{9}$$

where $Q[x] = g_3 x^2 + f_3 x + \alpha$ and

$$P[x] = (g_2^2 - g_1 g_3)x^4 - (g_1 f_3 + g_3 f_1 - 2g_2 f_2)x^3 + (f_2^2 - 4g_2 - f_1 f_3 - 2g_3 - g_1\alpha)x^2 - (2f_3 + f_1\alpha + 4f_2)x + (4 - 2\alpha). \tag{10}$$

The Lagrangian (9) describes a healthy excitation provided that two conditions are satisfied. First, to avoid ghosts, the time-derivative term must be positive definite. This puts a constraint on the parameter λ,

$$(3\lambda - 1)(\lambda - 1) > 0. \tag{11}$$

This condition can be easily fulfilled, e.g., by choosing $\lambda > 1$. Second, the dispersion relation of the mode ψ,

$$\omega^2 = \frac{\lambda - 1}{2(3\lambda - 1)}\frac{P[-p^2/M_P^2]}{Q[-p^2/M_P^2]}p^2, \tag{12}$$

yields the condition to avoid exponential instabilities [assuming that (11) holds],

$$P[x]/Q[x] > 0 \quad \text{at } x < 0. \tag{13}$$

This condition puts certain restrictions on the constants α, f_n, g_n. In particular, one gets

$$0 < \alpha < 2. \tag{14}$$

The precise form of the constraints on the other parameters coming from (13) is quite cumbersome and we prefer to omit it in this Letter. Nevertheless, the reader can easily convince him/herself that there is a nonempty region of the parameter space where (13) is satisfied.

In deriving (9) we have used in an essential way the dependence of the potential of the model on a_i. Indeed, in the absence of such dependence, as happens in the nonprojectable version of the original Hořava's proposal, the constants α, f_2, f_3, g_2, g_3 become zero and the polynomial $Q[x]$ vanishes identically. This means that the Lagrangian (9) is singular in this limit.

We can also compare the situation in our model with the projectable case of Hořava gravity. The latter is obtained from our expressions by taking the limit $\alpha \to \infty$, which forces ϕ to be constant in space. From the dispersion relation (12) one reads that in this case the scalar mode has an imaginary sound speed at low energies, cf. [11],

$$c_{\rm proj}^2 = -(\lambda - 1)(3\lambda - 1)^{-1} < 0.$$

This leads to an exponential instability, that can be tamed only by the higher order terms in the dispersion relation. Thus, the characteristic rate of the instability is of order $|c_{\rm proj}|M_P$. In principle, this rate can be suppressed by choosing $\lambda - 1$ to be extremely small. However, in this case the strong coupling scale of the theory becomes unacceptably low [7–9].

Let us return to our model. We should stress that the healthy behavior of the scalar mode can be achieved simultaneously with the stability in the helicity-2 perturbations. Indeed, the dispersion relation for the latter depends only on the coefficients in front of the operators in the first column of the list (7). After fixing these coefficients to ensure stability of the helicity-2 modes, we still have the freedom to choose the coefficients of the remaining operators in the list to satisfy (13).

The existence of a healthy quadratic action for the perturbations around Minkowski space-time guarantees the absence of short-scale instabilities for any smooth background. Indeed, at short scales a smooth metric can be approximated by the flat one, and the short-wavelength perturbations around this metric behave in the same way as in Minkowski. Besides, a regular quadratic Lagrangian allows us to develop the standard perturbation theory to account for the interactions of the modes. All the interaction terms are irrelevant with respect to the relativistic scaling (valid at low energies). This implies that their effect is negligible at low energies: their contributions grow with energy. This growth "halts" at the scale where the higher-derivative terms become important, as then the anisotropic scaling (4) prevails and all the interactions are at most marginal. Thus in the UV the theory remains weakly coupled provided that (i) the higher-derivative terms appear before any interactions become strong and (ii) the marginal couplings are small. Hence, with an appropriate choice of parameters one obtains a model which is weakly coupled up to trans-Planckian energies [12] (see [13]). Thus, the concern of [14] that strong coupling persists in the present model is unfounded.

Finally, the canonical structure of the model at hand does not present any problem. Indeed, whenever $\alpha \neq 0$ the lapse N is no longer a Lagrange multiplier, and the Hamiltonian constraint (obtained as the variation of the action with respect to N, $\mathcal{H} \equiv \frac{\delta S}{\delta N} = 0$) together with the equation $\pi_N = 0$ form a pair of second-class constraints (they have a nonvanishing Poisson bracket), which can be used to eliminate N and π_N. Hence, the pathologies [6,15] present in Hořava's original proposal do not appear.

Phenomenology.—At low energies the dispersion relation (19) for the scalar mode becomes linear,

$$\omega^2 = \frac{\lambda - 1}{3\lambda - 1}\left(\frac{2}{\alpha} - 1\right)p^2. \tag{15}$$

Depending on the values of λ and α, the propagation velocity of the scalar may differ from 1 (the velocity of the helicity-2 modes), making manifest the breakdown of Lorentz invariance down to low energies. The presence of a gapless scalar gravitational mode potentially means an interesting phenomenology. Leaving a detailed study for the future [9], let us perform a preliminary analysis.

First, we consider the large distance behavior of the gravitational field of a static pointlike source of mass m. Note that only the scalar part of the metric is excited in this case. The corresponding low-energy Lagrangian is obtained by combining the first two lines of Eq. (8) with the source term. The static part of the Lagrangian is

$$\mathcal{L} = \frac{M_P^2}{2}[-2\psi\Delta\psi + 4\phi\Delta\psi + \alpha(\partial_i\phi)^2] - m\phi\delta^3(\mathbf{x}). \tag{16}$$

The equations of motion following from (16) imply,

$$\phi = \psi = -\frac{m}{8\pi M_P^2(1 - \alpha/2)|\mathbf{x}|}. \tag{17}$$

Remarkably, the gravitational field has the same form as in GR with the effective Newton's constant

$$G_N^{-1} = 8\pi M_P^2(1 - \alpha/2). \tag{18}$$

In particular, the relation (17) implies that, in contrast to the case of Lorentz-invariant scalar-tensor theories of gravity [16], the deflection of light by the gravitational field in our model is the same as in GR.

The second phenomenological aspect that we discuss is low-energy cosmology. Notice that for the spatially homogeneous metric ansatz the proper acceleration a_i vanishes. As a consequence, the evolution of the Universe is insensitive to the terms with a_i in the action and differs at large distances from the case of GR only due to the presence of the parameter λ. Substituting the Friedman-Robertson-Walker metric into the action and varying with respect to the lapse one obtains the standard Friedmann equation $H^2 = (8\pi/3)G_{\rm cosm}\rho$, where H is the Hubble parameter and ρ is the total density of the Universe. The effective gravitational constant is

$$G_{\rm cosm}^{-1} = 4\pi M_P^2(3\lambda - 1). \tag{19}$$

Note that $G_{\rm cosm} \neq G_N$. A similar discrepancy between the gravitational constants appearing in the Newton's law and in the Friedmann equation also arises in certain low-energy theories constructed to describe the Lorentz-violating effects in gravity. These models include the Einstein-aether theory (see [17] for a recent review) and the gauged ghost condensate [18]. The observational bound on this discrepancy comes from the measurement of the primordial abundance of He^4 and reads [17,19] $|G_{\rm cosm}/G_N - 1| \lesssim 0.13$. In our model this implies rather mild constraints on the parameters α, $\lambda - 1 \lesssim 10^{-1}$.

The analogy with Einstein-aether theories is extremely useful to derive other phenomenological predictions. To make it explicit, it is convenient to consider the covariant form of our model, which can be obtained following the method developed in [7]. The foliation structure is encoded into a scalar field φ with a timelike gradient, the foliation slices are identified as the surfaces $\varphi = \text{const}$ and the quantities appearing in the action (1) with appropriate geometrical characteristics of these surfaces. In this way we obtain the covariant (diffeomorphism invariant) form of the low-energy part of the model action,

$$S_{\text{low-}E} = -\frac{M_P^2}{2}\int d^4x\sqrt{-g}\{^{(4)}R + (\lambda - 1)(\nabla_\mu u^\mu)^2 + \alpha u^\mu u^\nu \nabla_\mu u^\rho \nabla_\nu u_\rho\}, \quad (20)$$

where $u_\mu \equiv \nabla_\mu \varphi(\nabla_\nu \varphi \nabla^\nu \varphi)^{-1/2}$ and we have kept only terms with the smallest number of derivatives of u_μ. This action resembles a version of the Einstein-aether theory, with the additional constraint on the unit vector u_μ to be hypersurface orthogonal. From (20) it is transparent that the corrections to GR arising in the present model are due to the Lorentz-violating scalar field φ (entering into the action through u_μ). These effects are proportional to the parameters α, $\lambda - 1$. The analogy with the Einstein-aether theory indicates that the strongest observational bound arises from Solar-System tests of preferred-frame effects. Specifically, the post-Newtonian parameter α_2^{PPN} is constrained to be smaller than 4×10^{-7} [16]. Assuming α and $\lambda - 1$ to be of the same order, one expects that $\alpha_2^{\text{PPN}} \sim \alpha$, $\lambda - 1$ [13], as confirmed by explicit computation [9]. This yields the bound α, $\lambda - 1 \lesssim 4 \times 10^{-7}$. The latter bound together with the requirement of weak coupling implies an upper limit on the scale suppressing higher-derivative terms in the action of order 10^{15} GeV [13].

Discussion.—In this Letter we have described a natural extension of the nonprojectable version of Hořava's proposal for quantum gravity, which is free from the pathologies present in the original formulation. The extension is obtained by including in the action all terms allowed by the symmetries and the requirement that the model is power-counting renormalizable. It remains to be seen if the model provides a valid theory of quantum gravity.

At low energies the model reduces to a Lorentz-violating scalar–tensor theory. This potentially implies a rich low-energy phenomenology to be confronted with existing tests of GR. Remarkably, the effects of the scalar mode at large distances can be made weak by an appropriate choice of parameters without spoiling the good features of the model. Clearly, a detailed phenomenological study of the theory is needed to get better constraints.

A problem of any theory with high-energy breaking of Lorentz symmetry is the mechanism to recover it in the infrared. This issue arises because the Lorentz violation in the UV generically translates at low energies into different limiting propagation velocities for different particle species [20] (see [21] for a study in the present context), which are tightly constrained experimentally [22]. This seems to require a very precise fine-tuning of parameters to reconcile the theory with experiment. An elegant solution would be to find some (super)symmetry relating all the matter species in the UV and broken at a scale much lower than the characteristic scale of Lorentz violation. In this case, the Lorentz-violating effects would be suppressed at low energies by the ratio of the two scales. Clearly, this remains an open issue.

At a more theoretical level, the violation of Lorentz invariance down to the infrared leads to apparent paradoxes even in the absence of matter fields. We have seen that the propagation velocity of the scalar gravitational waves is in general different from that of the helicity-2 modes. This opens up the possibility to realize a perpetuum mobile of the second kind, and hence to violate unitarity, in gedanken processes involving black holes [23,24]. It will be interesting to see if and how this puzzle is resolved in the model proposed in this Letter.

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