Scalar Curvature of a Causal Set

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A one parameter family of retarded linear operators on scalar fields on causal sets is introduced. When the causal set is well approximated by 4 dimensional Minkowski spacetime, the operators are Lorentz invariant but nonlocal, are parametrized by the scale of the nonlocality, and approximate the continuum scalar D'Alembertian $\Box$ when acting on fields that vary slowly on the nonlocality scale. The same operators can be applied to scalar fields on causal sets which are well approximated by curved spacetimes in which case they approximate $\Box - \frac{1}{2}R$ where R is the Ricci scalar curvature. This can used to define an approximately local action functional for causal sets.

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The coexistence of Lorentz symmetry and fundamental, Planck scale spacetime discreteness has its price: one must give up locality. Since, if our spacetime is granular at the Planck scale, the "atoms of spacetime" that are nearest neighbors to a given atom will be of order one Planck unit of proper time away from it. The locus of such points in the approximating continuum Minkowski spacetime is a hyperboloid of infinite spatial volume on which Lorentz transformations act transitively. The nearest neighbors will, loosely, comprise this hyperboloid and so there will be an infinite number of them. Where curvature limits Lorentz symmetry, it may render the number of nearest neighbors finite but it will still be huge so long as the radius of curvature is large compared to the Planck length. Causal set theory is a discrete approach to quantum gravity which embodies Lorentz symmetry [1,2] and exhibits nonlocality of exactly this form [3,4].

Nonlocality looks to be simultaneously a blessing and a curse in tackling the twin challenges that any fundamentally discrete approach to the problem of quantum gravity must face. These are to explain (1) how the fundamental dynamics picks out a discrete structure that is well approximated by a Lorentzian manifold and (2) why, in that case, the geometry should be a solution of the Einstein equations. This is often referred to as the problem of the continuum limit but in the context of a fundamentally discrete theory in which the discreteness scale is fixed and is not taken to zero but rather the observation scale is large, it is more accurately described as the problem of the continuum approximation.

Consider first the problem of recovering a continuum from a quantum theory of discrete manifolds. (We adopt this term following Riemann [5] and use it to refer to causal sets, simplicial complexes, graphs, or whatever discrete entities the underlying theory is based on.) Whenever a background principle or structure in a physical theory is abandoned in order to seek a dynamical explanation for that structure, the state we actually observe becomes a very special one amongst the myriad possibilities that then arise. The continuum is just such a background assumption. In giving it up, generally one introduces a space of discrete manifolds in which the vast majority have no continuum approximation. There will therefore be a competition between the entropic pull of the huge number of noncontinuum configurations—choose one uniformly at random and it will not look anything like our spacetime—and the dynamical law which must suppress the contributions of these nonphysical configurations to the path integral. The following general argument shows that a local dynamics for quantum gravity will struggle to provide the required suppression. Consider the partition function as a sum over histories in which the weight of each discrete manifold is e^{-S} where S is the real Wick rotated action. As we increase the observation scale, the sum will be over discrete manifolds with an increasing number, N, of atoms. If the action is local—which in a discrete setting translates to it being a sum over contributions from each atom—then it will grow no faster than N times some constant, α, and so each weight is no smaller than $e^{-\alpha N}$. If the number of discrete manifolds with N atoms grows faster than exponentially with N, and if the majority of these discrete manifolds are not continuumlike then they will overwhelm the partition function and the typical configuration will not have a continuum approximation. Even when the number of discrete manifolds is believed to grow exponentially, entropy can still trump dynamics as was seen in the lack of a continuum limit in the Euclidean dynamical triangulations program [6–9]. Causal dynamical triangulations do better, see, e.g., [10–13], by restricting the class of triangulations allowed in the sum.

In the case of causal sets, the number of discrete manifolds of size N grows as $e^{N^2/4}$ [14] and a local action would give causal set theory little chance of recovering the continuum. So the nonlocality of causal sets holds out hope that the theory has a continuum regime and indeed there exist physically motivated, classically stochastic dynamical models for causal sets [15] in which the entropically favored configurations almost surely do not occur and those that do exhibit an intriguing hint of manifoldlikeness [16].

However, nonlocality poses a danger when it comes to the second challenge of recovering Einstein's equations. If we assume that a discrete quantum gravity theory does have a 4 dimensional continuum regime, and if the theory is local and generally covariant, then the long distance physics will be governed by an effective Lagrangian, which is a derivative expansion in which all diffeomorphism invariant terms are present but higher derivative terms are suppressed by the appropriate powers of the Planckian discreteness length scale, l:

$$\frac{\mathcal{L}_{\rm eff}}{\sqrt{-g}\hbar} = a_0 l^{-4} + a_1 l^{-2} R + a_2 R^2 + \dots \qquad (1)$$

where R is the Ricci scalar, a_1 and a_2 are dimensionless couplings of order 1, and the dots denote further curvature squared terms as well as cubic and higher terms. The coefficient of the leading term, a_0, is also naturally of order 1 which would make it 120 orders of magnitude larger than its observed value. However, that would also produce curvature on Planckian scales and so would not be compatible with the assumption of a continuum approximation. In a discrete theory, the question of why the cosmological constant does not take its natural value is the same question as why there is a continuum regime at all and we must look to the fundamental dynamics for its resolution. Assuming there is a resolution and a continuum regime exists, locality and general covariance then pretty much guarantee Einstein's equations due to the natural suppression of the curvature squared and higher terms compared to the Einstein-Hilbert term.

So, Lorentz symmetry and discreteness together imply nonlocality, but nonlocality blocks the recovery of general relativity, and if causal sets were incorrigibly nonlocal, this would be fatal. Suppose, however, that the nonlocality were somehow limited to length scales shorter than a certain l_k, which could be much larger than the Planckian discreteness scale, l, but yet have remained experimentally undetected to date. There is already evidence that this is possible and indeed causal sets admit constructions that are local enough to approximate the scalar D'Alembertian operator in 2 dimensional flat spacetime [17,18]. We add to this evidence here by exhibiting a family of discrete operators that approximate the scalar D'Alembertian in 4 dimensional flat spacetime. Further, both the 2D and 4D operators, when applied to scalar fields on causal sets which are well described by curved spacetimes approximate $\Box - \frac{1}{2}R$, where R is the Ricci scalar curvature. We use this to propose an action for a causal set which is approximately local.

We recall that a causal set (or causet) is a locally finite partial order, i.e., it is a pair $(\mathcal{C}, \preceq)$ where $\mathcal{C}$ is a set and $\preceq$ is a partial order relation on $\mathcal{C}$, which is (i) reflexive $x \preceq x$, (ii) acyclic $x \preceq y \preceq x \Rightarrow x = y$, and (iii) transitive $x \preceq y \preceq z \Rightarrow x \preceq z$, for all $x, y, z \in \mathcal{C}$. Local finiteness is the condition that the cardinality of any order interval is finite, where the (inclusive) order interval between a pair of elements $y \preceq x$ is defined to be $I(x, y) := \{z \in \mathcal{C} | y \preceq z \preceq x\}$. We write $x \prec y$ when $x \preceq y$ and $x \neq y$. We call a relation $x \prec y$ a *link* if the order interval $I(x, y)$ contains only x and y: they are nearest neighbors.

Sprinkling is a way of generating a causet from a d-dimensional Lorentzian manifold $(\mathcal{M}, g)$. It is a Poisson process of selecting points in $\mathcal{M}$ with density ρ so that the expected number of points sprinkled in a region of spacetime volume V is ρV. This process generates a causet whose elements are the sprinkled points and whose order is that induced by the manifold's causal order restricted to the sprinkled points. We say that a causet $\mathcal{C}$ is well approximated by a manifold $(\mathcal{M}, g)$ if it could have been generated, with relatively high probability, by sprinkling into $(\mathcal{M}, g)$.

We propose the following definition of a discrete D'Alembertian, B, on a causet $\mathcal{C}$ that is a sprinkling, at density $\rho = l^{-4}$, into 4D Minkowski space $\mathbb{M}^4$. Let $\phi: \mathcal{C} \to \mathbb{R}$ be a real scalar field, then

$$B\phi(x) := \frac{4}{\sqrt{6} l^2}\Big[-\phi(x) + \Big(\sum_{y\in L_1} - 9\sum_{y\in L_2} + 16\sum_{y\in L_3} - 8\sum_{y\in L_4}\Big)\phi(y)\Big], \qquad (2)$$

where the sums run over 4 layers L_i, $i = 1, \dots, 4$,

$$L_i := \{y \in \mathcal{C}: y \prec x \quad \text{and} \quad n(x, y) = i + 1\} \qquad (3)$$

and $n(x, y) := |I(x, y)|$. So, for example, layer L_1 is the set of all elements y that are linked to x and as described above, they will be distributed close to a hyperboloid that asymptotes to the past light cone of x and is proper time l away from x. This sum will not in general be uniformly convergent if it is over the elements of a sprinkling into infinite $\mathbb{M}^4$ so we introduce an IR cutoff, $L \gg l$, by embedding $\mathcal{C}$ in $\mathbb{M}^4$ and summing over the finitely many elements sprinkled in the intersection of the causal past of x and a ball of radius L centered on x. The details of the calculation that shows why 4 layers are necessary in 4D will appear elsewhere; however, see [18] for an explanation of why 3 layers are needed in 2D and the conjecture that 4D will require 4 layers.

Now let ϕ be a real test field of compact support on $\mathbb{M}^4$. If we fix a point $x \in \mathbb{M}^4$ (which we always take to be included in $\mathcal{C}$) and evaluate $B\phi(x)$ on a sprinkling into $\mathbb{M}^4$, its expectation value in this process is given by

$$\begin{aligned}\bar{B}\phi(x) &:= \mathbb{E}(B\phi(x)) \\ &= \frac{4}{\sqrt{6} l^2}\Big[-\phi(x) + \frac{1}{l^4}\int_{y\in J^-(x)} d^4 y \phi(y) \\ &\quad \times e^{-\xi}\Big(1 - 9\xi + 8\xi^2 - \frac{4}{3}\xi^3\Big)\Big],\end{aligned} \qquad (4)$$

where $\xi := l^{-4} V(x, y)$, $V(x, y)$ is the volume of the causal interval between x and y and there is an implicit cutoff L, the size of the support of ϕ, on the integration range.

It can be shown that this mean converges, as the discreteness scale is sent to zero, to the continuum D'Alembertian of ϕ,

$$\lim_{l\to 0} \bar{B}\,\phi(x) = \Box\phi(x) \tag{5}$$

and that $\bar{B}\phi(x)$ is well approximated by $\Box\phi(x)$ when the characteristic length scale, λ, on which $\phi(x)$ varies is large compared to l. $\bar{B}$ is therefore effectively sampling the value of the field only in a neighborhood of x of size of order l and the mean, at least, of B is about as local as it can possibly be, given the discreteness.

To see roughly how this can happen, notice that the integrand in (4) is negligible for $\xi > \alpha^4$ where α is such that $e^{-\alpha^4} \ll 1$. The significant part of the integration range therefore lies between the past light cone of x and the hyperboloid $\xi = \alpha^4$ and comprises a part within a neighborhood of x of size αl—whence the local contribution—and the rest which stretches off far down the light cone. It is this second part of the range which threatens to introduce nonlocality but because it can be coordinatized by ξ itself and some coordinates η^a on the hyperboloid the integration over it will be proportional to

$$\int d^3\eta \int_0^{\alpha^4} d\xi e^{-\xi}\left(1 - 9\xi + 8\xi^2 - \frac{4}{3}\xi^3\right)\phi(\xi, \eta^a). \tag{6}$$

If ϕ is nearly constant over length scale αl, the ξ integration is close to zero and the contribution is suppressed.

The fluctuations in $B\phi(x)$, however, are a different matter: if the physical IR cutoff L is fixed and the discreteness scale sent to zero, i.e., the number of causet elements N grows, simulations show the fluctuations around the mean grow rather than die away and $B\phi(x)$ will not be approximately equal to the continuum $\Box\phi(x)$. To dampen the fluctuations we follow [18] and introduce an intermediate length scale $l_k \geq l$ and smear out the expressions above over this new scale, with the expectation that when $l_k \gg l$ the inhering averaging will suppress the fluctuations via the law of large numbers. Thus we seek a discrete operator, B_k, whose mean is given by (4) but with l replaced by l_k:

$$\bar{B}_k\phi(x) = \frac{4}{\sqrt{6}l_k^2}\left[-\phi(x) + \frac{1}{l_k^4}\int_{y\in J^-(x)} d^4y\,\phi(y) \times e^{-\xi}\left(1 - 9\xi + 8\xi^2 - \frac{4}{3}\xi^3\right)\right], \tag{7}$$

where now $\xi := l_k^{-4}V(x, y)$. Working back, one can show that the discrete operator, B_k, with this mean is

$$B_k\phi(x) = \frac{4}{\sqrt{6}l_k^2}\left[-\phi(x) + \epsilon\sum_{y\prec x} f(n(x, y), \epsilon)\phi(y)\right], \tag{8}$$

where $\epsilon = (l/l_k)^4$ and

$$f(n, \epsilon) = (1-\epsilon)^n\left[1 - \frac{9\epsilon n}{1-\epsilon} + \frac{8\epsilon^2 n!}{(n-2)!(1-\epsilon)^2} - \frac{4\epsilon^3 n!}{3(n-3)!(1-\epsilon)^3}\right]. \tag{9}$$

B_k reduces to B when $\epsilon = 1$. B_k effectively samples ϕ over elements in 4 broad bands with a characteristic depth l_k, the bands' contributions being weighted with the same set of alternating sign coefficients as in B. Since (7) is just (4) with l replaced by l_k, the mean of $B_k\phi(x)$ is close to $\Box\phi(x)$ when the characteristic scale over which ϕ varies is large compared to l_k. Now, however, numerical simulations show that the fluctuations are tamed. Points were sprinkled into a fixed causal interval in $\mathbb{M}^4$ between the origin and $t = 1$ on the t axis, at varying density $\rho = \frac{N}{V}$, where volume $V = \frac{\pi}{24}$. For each N, 100 sprinklings were done and for each sprinkling, $B_k\phi$ was calculated at the topmost point of the interval for $\phi = 1$ and $l_k = 0.16$. For $N = 5000$, the mean was $\mu = 9.35$ and the standard deviation $s.d. = 134.8$. For $N = 10\,000$, $\mu = 4.00$ and $s.d. = 102.6$ and for $N = 20\,000$, $\mu = 1.12$, and $s.d. = 58.8$. These results indicate that the fluctuations do die away, as anticipated, as N increases and are consistent with the dependence $N^{-(1/2)}$. Further results will appear elsewhere.

The operators B and B_k derived in both 2D (in [18]) and 4D are defined in terms of the order relation on $\mathcal{C}$ alone and so can be applied to a scalar field on *any* causet. If, therefore, $(\mathcal{M}, g)$ is a (2D or 4D) curved spacetime and ϕ is a scalar field on $\mathcal{M}$, we can compute $B_k\phi(x)$ on a sprinkling into $\mathcal{M}$ and calculate its mean. Let V_2 and V_4 be the volumes of the intervals in 2D and 4D, respectively, $\xi_2 := V_2(x, y)l_k^{-2}$ and $\xi_4 := V_4(x, y)l_k^{-4}$. Then, in the presence of curvature,

$$\bar{B}_k^{(2)}\phi(x) = \frac{2}{l_k^2}\left[-\phi(x) + \frac{2}{l_k^2}\int_{y\in J^-(x)} d^2y\sqrt{-g} \times e^{-\xi_2}\left(1 - 2\xi_2 + \frac{1}{2}\xi_2^2\right)\phi(y)\right] \tag{10}$$

and

$$\bar{B}_k^{(4)}\phi(x) = \frac{4}{\sqrt{6}l_k^2}\left[-\phi(x) + \frac{1}{l_k^4}\int_{y\in J^-(x)} d^4y\sqrt{-g} \times e^{-\xi_4}\left(1 - 9\xi_4 + 8\xi_4^2 - \frac{4}{3}\xi_4^3\right)\phi(y)\right], \tag{11}$$

in 2D and 4D, respectively.

These expressions can be evaluated using Riemann normal coordinates and in both cases we find

$$\lim_{l_k\to 0}\bar{B}_k^{(i)}\phi(x) = \left(\Box - \tfrac{1}{2}R(x)\right)\phi(x). \tag{12}$$

The limit is a good approximation to the mean when the field ϕ varies slowly over length scales l_k *and* the radius of curvature $r \gg l_k$.

If the damping of fluctuations found in simulations in flat space are indicative of what happens in curved space then, for a fixed large enough IR cutoff, L, the nonlocality length scale l_k can be chosen such that $l \ll l_k \ll L$ and the value of $B_k\phi$ for a single sprinkling will be close to the mean. If B_k is applied to the constant field $\phi = -2$, we therefore obtain an expression that is close to the scalar curvature of the approximating spacetime.

In each of 2D and 4D, we can now define a one parameter family of candidate actions, $S_k[\mathcal{C}]$, for a causal set, $\mathcal{C}$, by summing $B_k(-1)$ over the elements of $\mathcal{C}$, times $\hbar l^2$ to get the units right, times a number of order one, which in 4D is the ratio of l^2 to l_p^2, where $l_p = \sqrt{8\pi G\hbar}$ is the rationalized Planck length. When the nonlocality length l_k equals the discreteness length l, $B_k = B$ and the action, $S[\mathcal{C}]$ takes a particularly simple form as an alternating sum of numbers of small order intervals in $\mathcal{C}$. Up to factors of order one, we have in 2D and 4D, respectively:

$$\frac{1}{\hbar}S^{(2)}[\mathcal{C}] = N - 2N_1 + 4N_2 - 2N_3 \qquad (13)$$

and

$$\frac{1}{\hbar}S^{(4)}[\mathcal{C}] = N - N_1 + 9N_2 - 16N_3 + 8N_4, \qquad (14)$$

where N is the number of elements in $\mathcal{C}$ and N_i is the number of $(i+1)$ element inclusive order intervals in $\mathcal{C}$.

Because B is the most non-nonlocal of the operators in the family, the action $S[\mathcal{C}]$ is a sum of contributions each of which is not close to the value of the Ricci scalar at the corresponding point of the continuum approximation. However, one might expect that if the curvature is slowly varying on some intermediate scale, which we might as well call l_k, the averaging involved in the summation might perform the same role of suppressing the fluctuations as the smearing out of the operator itself so that the whole action $S[\mathcal{C}]$ is a good approximation to the continuum action when l_k is the appropriate size.

There are many new avenues to explore. Can we use these results to define a quantum dynamics for causal sets? In 2D is there a relation with the Gauss-Bonnet theorem? Can we analytically continue the action in an appropriate way [19] to enable Monte Carlo simulations of the path sum? What sort of phenomenology might emerge from such actions? To answer this latter question, we need to know how big l_k must be so that the action $S[\mathcal{C}]$ is a good approximation to the Einstein-Hilbert action of the continuum $S_{\rm EH}[g]$. In [18], a rough estimate is reported that in dimension 4, $l_k \gg (l^2L)^{1/3}$. Taking L to be the Hubble scale, that would mean that in the continuum regime, only spacetimes whose curvature was constant over a scale $(l^2L)^{1/3}$ would be able to have an approximately local fundamental action. One might expect therefore that the phenomenological IR theory of gravity that could emerge from such a fundamental theory would be governed by an effective Lagrangian

$$\frac{\mathcal{L}_{\rm eff}}{\sqrt{-g}\hbar} = b_0 l_k^{-4} + b_1 l_k^{-2}R + b_2R^2 + \ldots \qquad (15)$$

where b_1 and b_2 are of order 1, b_0 is set to its observed value, and where l_k varies with epoch and today is much larger than the Planck scale. The phenomenological implications of these ideas remain to be explored.

We end by pointing out that these results have a relevance beyond causal set theory as they provide a "proof of concept" for the mutual compatibility of Lorentz invariance, fundamental spacetime discreteness, and approximate locality.

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