Implementing Arbitrary Phase Gates with Ising Anyons

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Ising-type non-Abelian anyons are likely to occur in a number of physical systems, including quantum Hall systems, where recent experiments support their existence. In general, non-Abelian anyons may be utilized to provide a topologically error-protected medium for quantum information processing. However, the topologically protected operations that may be obtained by braiding and measuring topological charge of Ising anyons are precisely the Clifford gates, which are not computationally universal. The Clifford gate set can be made universal by supplementing it with single-qubit $\pi/8$-phase gates. We propose a method of implementing arbitrary single-qubit phase gates for Ising anyons by running a current of anyons with interfering paths around computational anyons.

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Non-Abelian anyons, quasiparticles with exotic exchange statistics described by multidimensional representations of the braid group [1], can provide fault-tolerant platforms for quantum computation. Their nonlocal state space can be used to encode qubits that are impervious to local perturbations. Topologically protected computational gates may be implemented by braiding the anyons [2] or measuring their topological charge [3].

Ising-type anyons currently appear to be the most likely platform on which topological quantum computation will be actualized. They are expected to occur in a number of systems, including 2nd Landau level quantum Hall states [4,5], $p_x + ip_y$ superconductors [6], lattice models [7], topological insulator-superconductor interfaces [8], and any generic 2D system with Majorana fermions [9]. Their existence in the $\nu = 5/2$ quantum Hall state is supported by recent experiments [10,11].

The braiding transformations of Ising anyons are given by the spinor representations of SO$(2n)$ [12]. The set of gates that may be obtained through braiding and/or topological charge measurement of Ising anyons is encoding dependent, but never computationally universal. For the standard qubit encoding (i.e., one qubit in four anyons), the computational gates obtained via braiding or measurement of anyon pairs are the single-qubit Clifford gates. These gates can be generated by the Hadamard and $\pi/4$-phase gates, where $R(\theta) = \text{diag}[1, e^{i\theta}]$ is called the "$\theta/2$-phase gate."

The CNOT gate may be implemented by allowing the use of nondemolitional measurements of the collective topological charge of four anyons [13,14]. Adding this generates the full set of Clifford gates, which can be efficiently simulated on a classical computer, but becomes universal when supplemented with a $\pi/8$-phase gate [15].

One way to obtain $\pi/8$-phase gates (as well as CNOT gates) is through dynamical topology change of the system [16]. However, this requires complicated physical manipulations of the system which are currently infeasible, such as switching between planar and nonplanar geometries.

Alternatively, if one can implement ideal (e.g., topologically protected) Clifford gates, then they can be used to perform "magic state distillation" [13,14] to produce error-corrected $\pi/8$-phase gates from noisy ones. This purification protocol (which has polylog overhead) consumes several copies of a magic state, e.g., $|A_{\pi/4}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\pi/4}|1\rangle)$, and outputs a single qubit with higher polarization along a magic direction. Once a sufficiently pure magic state is produced, it may then be consumed to generate a $\pi/8$-phase gate. This protocol permits a remarkably high error threshold of over 0.14 for the noisy gates, as compared to the "high" threshold of 10^{-3} for postselected quantum computation [17]. Hence, it is important to devise practical methods of generating the $\pi/8$-phase gate within this error threshold for systems with Ising anyons.

A simple proposal for this is to move bulk quasiparticles close enough to each other to let the microscopic physics split the energy degeneracy of the fusion channels encoding a qubit. The resulting time evolution can produce arbitrary phase gates, albeit unprotected ones in need of error correction (e.g., by magic state distillation). However, the energy splitting caused by bringing two quasiparticles together oscillates rapidly with their separation [18,19], so small errors in the quasiparticles' spatial separation will translate into large errors in the phase. Thus, this approach appears unlikely to be able to meet even the generous error threshold of magic state distillation.

In this Letter, we propose a method of implementing arbitrary phase gates for systems with Ising-type anyons that aims to be more practical and to achieve a manageable error rate. This method involves a device consisting effectively of a beam splitter or tunneling junction that is used to run a current of anyons through interfering paths around computational anyons. We first analyze the effect of such a device using a semiclassical picture of the anyonic current applicable to general Ising systems. Subsequently, we perform a more detailed analysis (including error estimates) for Ising systems in which the anyonic current is provided by edge modes described by conformal field theory.

For the purpose of constructing the phase gate, we consider a topological qubit encoded in a pair of anyons carrying Ising topological charge σ. The two possible fusion channels I and ψ correspond to the qubit basis states $|0\rangle$ and $|1\rangle$. In the quantum Hall context, the anyons comprising the topological qubits may be localized using quantum antidots, each carrying a topological charge of σ (i.e., an odd number of $e/4$ quasiparticles), as proposed in [20]. The two anyons comprising a qubit are placed in a "sack" geometry as shown in Fig. 1. This geometry was proposed for the detection of non-Abelian statistics in [21], where it was described as a "wormhole." The sack may be created by deforming the edge of the quantum Hall droplet. The current flowing around the edge can tunnel from $-a/2$ to $a/2$ (so the sack has perimeter length a) with a strength determined by the distance d across the constriction. In the weak-tunneling, low temperature, low voltage limit, the quasiparticles with the most relevant tunneling operators will dominate the tunneling current. For Ising-type quantum Hall states, this will generally include but not be restricted to the fundamental quasiholes that carry charge σ. However, quasiparticles that do not carry σ will have no effect on the topological qubit here, so we will neglect them in our analysis. As we will see, the interference between the possible trajectories from left to right enacts a nontrivial transformation on the qubit.

In other possible physical realizations of Ising anyons, the σ anyons comprising a qubit may need to be pinned by other means; e.g., in a chiral p-wave superconductor, a hole may be bored through the sample where flux can be trapped. It may be also be easier in some realizations to construct interfering paths for a beam of bulk quasiparticles rather than to rely on edge quasiparticles. With this situation in mind, we now compute semiclassically the effect of a beam of σ quasiparticles incident from the left. This calculation will also capture some of the features of the more involved edge theory calculation, relevant to the quantum Hall setting. For ease of comparison with Fig. 1, we use terminology appropriate to that picture. We assume that at the tunneling junction in Fig. 1 a σ quasiparticle can tunnel with amplitude $\mathcal{T}$ and will continue along the edge with amplitude $\mathcal{R}$.

We can treat the anyons semiclassically and analyze the effect that sending them through the device has on the qubit. Braiding statistics contributes a factor of $+1$ or -1 when a σ anyon travels one full circuit around a region containing topological charge I or ψ, respectively. This non-Abelian contribution is in addition to the Abelian phase α acquired when a σ anyon encircles the device loop counterclockwise. This phase α contains the Abelian

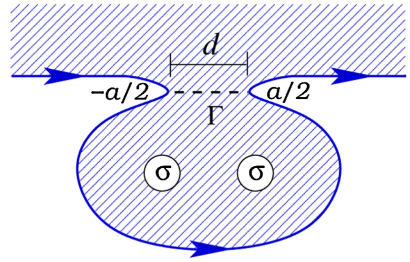

FIG. 1 (color online). An implementation of the phase gate device in Ising-type quantum Hall states. A section of the Hall fluid (hatched region) is formed into a sacklike enclosure around two σ anyons encoding a qubit. The edge current (arrowed lines) tunnels quasiparticles across the constriction with strength Γ, inducing a phase gate on the qubit.

statistical angle, the Aharanov-Bohm phase, and possibly other terms, depending on the realization of the device.

The resulting transformation to the qubit when one σ anyon has passed through the device is

$$U = \mathcal{T}e^{-i\alpha}\boldsymbol{\sigma}_z + |\mathcal{R}|^2 \sum_{n=0}^{\infty}(-\mathcal{T}^* e^{i\alpha}\boldsymbol{\sigma}_z)^n = \begin{bmatrix} \frac{1+\mathcal{T}e^{-i\alpha}}{1+\mathcal{T}^* e^{i\alpha}} & 0 \\ 0 & \frac{1-\mathcal{T}e^{-i\alpha}}{1-\mathcal{T}^* e^{i\alpha}} \end{bmatrix}, \quad (1)$$

where $\boldsymbol{\sigma}_z$ accounts for the non-Abelian braiding statistics. Here the first term results from direct tunneling across the constriction, and the remaining terms describe the effect of the σ quasiparticle passing around the edge of the sack one or many times. This does not transfer topological charge to the qubit, so the matrix U is diagonal and unitary. However, braiding a σ quasiparticle from the beam around the computational σ anyons is topologically equivalent to processes that transfer topological charge ψ between the computational pair. These are the same processes that would cause energy splitting between the otherwise degenerate fusion channels of the qubit when its σ quasiparticles are brought close together [22]. Hence, the net effect of passing a σ through the device is similar to that of splitting the energy, i.e., to produce a relative phase between these channels. Up to an overall phase, $U = R(\theta)$, where $\theta = 2\arctan[\frac{2|\mathcal{T}|\sin\gamma}{1-|\mathcal{T}|^2}]$, and $\gamma = \alpha - \arg\{\mathcal{T}\}$. For $|\mathcal{T}| \ll 1$, this gives $\theta \simeq 4|\mathcal{T}|\sin\gamma$. The phase gate generated using this device may be controlled by sending multiple σ quasiparticles through the system or by adjusting the experimental variables $\mathcal{T}$ and α.

For Ising-type systems that support an anyonic edge current, such as those in Refs. [4–9], we should go beyond this semiclassical calculation and analyze the quasiparticle tunneling and interference using the proper edge theory. The combined edge and qubit system is described by the Hamiltonian $H = H_E \otimes \mathbb{1} + H_{\text{tun}}(t) \otimes \boldsymbol{\sigma}_z$, where H_E is the Hamiltonian describing the unperturbed edge and H_{tun} describes tunneling of σ quasiparticles across the constriction. As before, the $\boldsymbol{\sigma}_z$ represents the braiding statistics of the edge σ with the qubit, picking up a minus sign each time the σ braids around the ψ charge. The strength of the tunneling Hamiltonian can be adjusted by changing the separation distance d across the sack constriction. We represent the density matrix of the combined system by χ and the qubit's density matrix is obtained from this by tracing out the edge $\rho = \mathrm{Tr}_E \chi$.

Solving the interaction picture Schrödinger equation

$$i\frac{d\tilde{\chi}(t)}{dt} = [\tilde{H}_{\mathrm{tun}}(t) \otimes \boldsymbol{\sigma}_z, \tilde{\chi}(t)], \tag{2}$$

where $\tilde{A}(t) = e^{iH_E(t-t_0)}A(t)e^{-iH_E(t-t_0)}$, with the assumption that the edge and qubit are unentangled at time $t = t_0$, we find that

$$\rho(t) = \begin{bmatrix} \rho_{00}(t_0) & e^{-\varsigma^2/2}e^{-i\theta}\rho_{01}(t_0) \\ e^{-\varsigma^2/2}e^{i\theta}\rho_{10}(t_0) & \rho_{11}(t_0) \end{bmatrix}, \tag{3}$$

where θ and ς^2 are real-valued time dependent quantities, and $\varsigma^2 \geq 0$. The diagonal elements of the qubit density matrix are unaltered from their initial state, as the Hamiltonian commutes with $\mathbb{1} \otimes \boldsymbol{\sigma}_z$. The sack geometry will therefore implement a phase gate $R(\theta)$ with phase-damping noise parametrized by ς^2.

Applying a Hadamard gate and then this noisy phase gate to $|0\rangle$ creates a magic state $|A_{\pi/4}\rangle$ with error

$$\epsilon = 1 - \langle A_{\pi/4}|\rho|A_{\pi/4}\rangle = \frac{1}{2}\left[1 - e^{-\varsigma^2/2}\cos\left(\theta - \frac{\pi}{4}\right)\right]. \tag{4}$$

If $\epsilon < 0.14$, this can be used with magic state distillation to generate an error-corrected $\pi/8$-phase gate [13,14].

Computing the values of θ and ς^2 to second order in the tunneling Hamiltonian, we have

$$\theta \simeq 2\int_{t_0}^{t} dt' \langle \tilde{H}_{\mathrm{tun}}(t')\rangle, \tag{5}$$

$$\varsigma^2 \simeq -\theta^2 + 4\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \langle \tilde{H}_{\mathrm{tun}}(t_1)\tilde{H}_{\mathrm{tun}}(t_2)\rangle. \tag{6}$$

We note that ς^2 takes the form of a variance in the phase.

To compute concrete values of θ and ς^2 for the most physically relevant example, we turn to the field theoretic description of the edge of a Moore-Read quantum Hall state [4]. The Lagrangian for the unperturbed edge is [23]

$$\mathcal{L}_E = \frac{1}{2\pi}\partial_x\varphi(\partial_t + v_c\partial_x)\varphi + i\psi(\partial_t + v_n\partial_x)\psi, \tag{7}$$

where the charged and neutral sectors are, respectively, described by the chiral boson (φ) and fermion (ψ) modes, with velocities v_c and v_n. The operator that tunnels charge $e^* = e/4$ σ quasiparticles across the constriction is

$$H_{\mathrm{tun}} = \Gamma e^{-i\alpha}\sigma\left(\frac{a}{2}\right)\sigma\left(-\frac{a}{2}\right)e^{i\varphi(a/2)/\sqrt{8}}e^{-i\varphi(-a/2)/\sqrt{8}} + \mathrm{H.c.}, \tag{8}$$

where α includes the Aharanov-Bohm phase (e^*BA) acquired in traveling around the sack as well as any Abelian braiding statistics factors.

Assuming the edge was initially in thermal equilibrium at temperature T, i.e. $\chi(t_0) = (e^{-H_E/T}/\mathrm{Tr}_E[e^{-H_E/T}]) \otimes \rho(t_0)$, we find

$$\langle \tilde{H}_{\mathrm{tun}}(t)\rangle = 2\left(\frac{\lambda\pi T/v_c}{\sinh\frac{\pi T a}{v_c}}\right)^{g_c}\left(\frac{\lambda\pi T/v_n}{\sinh\frac{\pi T a}{v_n}}\right)^{g_n}|\Gamma|\sin\gamma. \tag{9}$$

Here λ is a short-range cutoff, $g_c = 1/8$ and $g_n = 1/8$ are the scaling exponents of the charge and neutral modes, respectively, and $\gamma = \alpha - \arg\{\Gamma\} + \pi/2$.

From Eqs. (5) and (9), we see that there are several experimental parameters which may be used to control the phase θ generated using the sack geometry. In particular, we envision d and the area A enclosed in the sack as the primary physical quantities to adjust, since these provide a practical means of tuning $|\Gamma|$ and γ, respectively, while keeping the other quantities essentially constant. With a properly designed geometry, these quantities can be adjusted sufficiently while causing only negligible changes to a. In contrast to the tunneling amplitude of neutral ψ excitations, which oscillates rapidly with distance [18,19] (and can be understood as Friedel oscillations in a composite fermion picture), the tunneling amplitude of σ quasiparticles does not oscillate and decays as $\Gamma \sim e^{-(e^*d/2e\ell_B)^2}$ for $d \gg \ell_B$ (the magnetic length) [11,24].

There are several ways to adjust the phase γ for quantum Hall systems. One practical method is to alter the total area enclosed in the sack by using a side gate. This leads to a change in the flux enclosed in the two interfering current paths, and thus a change in the Aharanov-Bohm phase included in γ. Another method for changing γ is by applying a current along the edge of the system. This may be implemented via a voltage difference between the edge that forms the sack structure and the edge on the other side of the electron gas. Driving this current populates or depopulates charge on the edge of the electron gas, and hence changes the area as a side gate would.

Let us hold fixed all the experimental parameters except the tunneling amplitude, which we vary as $\Gamma(t) = \Gamma_0 f(t)$, for $f(t)$ a general (real, non-negative) signal profile with characteristic "duration" time scale $\tau \equiv \int_{-\infty}^{\infty} dt f(t)$. This gives $\theta \simeq \omega\tau$, where $\omega \equiv 2\langle \tilde{H}_{\mathrm{tun}}(t)\rangle/f(t)$, and

$$\varsigma^2 \simeq \frac{\omega^2}{\sin^2\gamma}\int_{-\infty}^{\infty} dt \eta(t)F(t), \tag{10}$$

$$\eta(t) = \frac{1}{2}(\Upsilon_c^{g_c}\Upsilon_n^{g_n-1/4} - \Upsilon_c^{-g_c}\Upsilon_n^{-g_n})\sqrt{\frac{1+\Upsilon_n^{1/2}}{2}} + \left(\Upsilon_c^{-g_c}\Upsilon_n^{-g_n}\sqrt{\frac{1+\Upsilon_n^{1/2}}{2}} - 1\right)\sin^2\gamma, \tag{11}$$

$$\Upsilon_{c,n}(t) = 1 - \frac{\sinh^2(\frac{\pi T a}{v_{c,n}})}{\sinh^2(\pi T t - i\delta)}, \tag{12}$$

where $F(t) \equiv \int_{-\infty}^{\infty} dt' f(t')f(t'-t)$. We note that $\eta(t) \to 0$

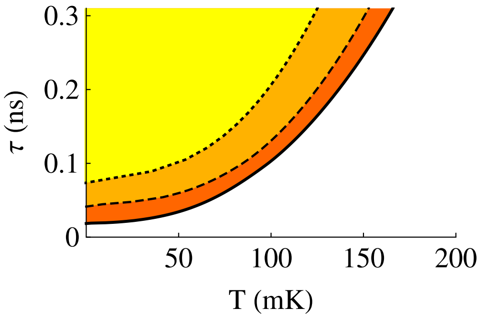

FIG. 2 (color online). Error-correction threshold curves for implementing a $\pi/8$-phase gate using the sack geometry in a Moore-Read state, when $\gamma = \pi/2$ (solid curve), $\pi/8$ (dashed curve), and $\pi/16$ (dotted curve). Magic state distillation is applicable in the shaded region above these curves (which can thus be viewed as indicating the minimum required gate duration as a function of temperature).

exponentially for long times with a decay rate proportional to the temperature. We generally have the bound

$$\varsigma^2 \lesssim \frac{\omega^2}{\sin^2\gamma} \frac{\sinh(\frac{\pi T a}{v_n})}{\pi T} \kappa\tau, \tag{13}$$

where κ is a dimensionless function of g_c, g_n, and $\frac{\sinh(\pi T a/v_c)}{\sinh(\pi T a/v_n)}$. When $\frac{a}{v_{c,n}} \ll \frac{1}{\pi T}$ (i.e., a is much shorter than the thermal coherence length), this becomes a temperature independent bound $\varsigma^2 \lesssim \frac{\omega^2 a/v_n}{\sin^2\gamma}\kappa\tau$ with κ now depending only on g_c, g_n, and v_n/v_c. Using $\omega \simeq \theta/\tau$ in these expressions, we see that it is favorable to increase the duration τ (e.g., by using weaker tunneling) used to enact a particular phase gate, since the bound decreases as $1/\tau$. However, this must be balanced with the need to keep time scales much shorter than the qubits' coherence time.

To demonstrate that the $\pi/8$-phase gate can (at least in principle) be implemented with sufficiently low error using this device, we compute ς resulting for $\theta = \pi/4$ using a sack of length $a = 1\ \mu$m, a rectangular pulse of duration τ, i.e., $f(t) = 1$ for $0 < t < \tau$ and 0 elsewhere, and velocities $v_c = 10^5$ m/s and $v_n = 10^4$ m/s estimated for the $\nu = 5/2$ state from numerical studies [25,26]. In Fig. 2, we display the resulting region of parameter space (for different values of γ) in which the error is below the threshold $\epsilon < 0.14$ for magic state distillation. The threshold curves move up as γ is varied away from $\pi/2$, and will diverge as $\gamma \to 0$ or π.

It is straightforward to repeat the preceding edge theory analysis for other Ising-type systems with a conformal edge theory. The results are again given by the preceding equations, but with different values of g_c, g_n, e^*, and α. The values of these quantities for Ising-type quantum Hall candidates [4,5] for all the observed 2nd Landau level plateaus can be found in [11]. For systems with chargeless Ising edges [6–9], one has $g_c = e^* = 0$ and $g_n = 1/8$.

In addition to offering a correctable error rate, this phase gate implementation offers several practical advantages. The computational anyons may remain stationary while only the edge of the system is manipulated, thus circumventing the need for fine control over the motion of bulk quasiparticles and making it compatible with "measurement-only" proposals [3]. As this device would only require the use of established techniques for deforming the edge using top and side gates [10], it provides the first realistic proposal for achieving universal quantum computation using Ising anyons.

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