Entanglement Gap and a New Principle of Adiabatic Continuity

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We give a complete definition of the entanglement gap separating low-energy, topological levels from high-energy, generic ones, in the "entanglement spectrum" of fractional quantum Hall (FQH) states. This is accomplished by removing the magnetic length inherent in the FQH problem—a procedure which we call taking the conformal limit. The counting of the low-lying entanglement levels starts off as the counting of modes of the edge theory of the FQH state, but quickly develops finite-size effects which we find to serve as a fingerprint of the FQH state. As the sphere manifold where the FQH resides grows, the level spacing of the states at the same angular momentum goes to zero, suggestive of the presence of *relativistic* gapless edge states. By using the adiabatic continuity of the low-entanglement energy levels, we investigate whether two states are topologically connected.

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Topological phases of matter generally lack local order parameters that can distinguish them from trivial ones. Moreover, extracting the topological order directly from the ground-state wave function is a nontrivial task. For incompressible states, several nonlocal indicators of the topological nature of the phase, such as ground-state degeneracy on compact high genus manifolds, the structure of edge modes and their scaling exponents, as well as quantum dimension analysis exist, but still do not fully describe the topological phase. The measure of choice has so far been the entanglement entropy (EE), especially its topological part [1,2]. For a given state $|\Psi_0\rangle$ and according density matrix $\rho = |\Psi_0\rangle\langle\Psi_0|$, let the Hilbert space be decomposed as a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Defining $\rho_A \equiv \mathrm{Tr}_B[\rho]$, the EE with respect to the partitioning (A, B) is defined by $S_A = -\mathrm{Tr}_A[\rho_A \ln\rho_A]$. For two-dimensional quantum systems, except in special cases where analytical solutions can be found [3], extracting the topological part of the EE becomes a highly nontrivial (and almost impossible) task.

While the EE is just one number, it was recently proposed and numerically substantiated [4] that the entanglement spectrum (ES), i.e., the full set of eigenvalues of ρ_A, understood as a geometric partition of the quantum Hall sphere [5], is a better indicator of topological order in the ground state of fractional quantum Hall (FQH) systems. Writing the eigenvalues as the spectrum of a fictitious Hamiltonian, $\rho_A = \exp(-H)$, where one can think of the H eigenvalues ξ as a quasienergy (or entanglement energy), Li and Haldane [4] showed that the low quasienergy spectrum for generic gapped $\nu = 5/2$ states exhibits a universal structure, related to conformal field theory (CFT). A few of the eigenvalues displaying this CFT counting are separated from a nonuniversal high-energy spectrum by an entanglement gap which was conjectured to be finite in the thermodynamic (TD) limit [4]. This gap itself was proposed as a "fingerprint" of the topological

order present. It was subsequently shown that the ES can meaningfully distinguish among states which have similar finite-size overlap with each other, but different edge structures [6]. Recently, the ES was found to detect topological order in gapless spin chains [7].

An unambiguous definition of the entanglement gap is still an open question. As the angular momentum of the northern hemisphere grows, i.e., L_z^A, the z component of the angular momentum of the complementary region A, is reduced, the entanglement gap collapses: in finite sizes, and for good incompressible states with large gap, this happens at roughly 4–5 momenta below that of the minimum angular momentum for the hemisphere where the cut was made [see Fig. 1(a)]. For these 4–5 momenta, the state shows the counting of the edge modes of its corresponding CFT [4], and deviates from this counting once the entanglement spectrum "feels" the edge (north pole) of the system. For other FQH states (such as the $\nu = 5/2$ Pfaffian state), the entanglement gap as currently defined is rather small and already disappears after 2–3 angular momenta above the minimum one. As we raise the angular momentum of the northern hemisphere, the ES levels form a continuum of states, which previously led to the conclusion that these levels are not useful for determining the character of a FQH state. Also, if we assume the conjectured mapping of entanglement energies to edge mode energies, it is unclear why states at the same angular momentum would have different entanglement energies, as the dispersion on the edge is relativistic.

In this Letter, we give a precise definition of the entanglement gap. We notice that the previous applications of the ES [4,6] contained the geometry of the Landau orbitals on the manifold in question (sphere), and hence implicitly had involved the magnetic length. Inspired by our previous findings on spin chains [7], by removing the magnetic length from the problem, we obtain the "conformal limit" (CL) of the FQH polynomial. For model FQH states, the
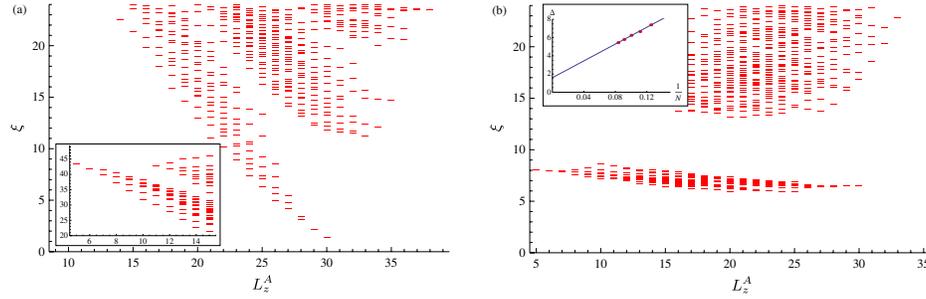
FIG. 1 (color online). Entanglement spectrum for the $N = 11$ bosons, $N_\phi = 20$, $\nu = 1/2$ Coulomb state on the sphere. The cut is such that $l_A = 10$ orbitals and $N_A = 5$ bosons. (a) Standard normalization on the quantum Hall sphere. The inset show the remainder part of the spectrum where the entanglement levels exceed $\xi = 24$. (b) CL normalization. We observe that the CL separates a set of universal low-lying energy states, which allows an unambiguous definition of the entanglement gap over all L_z^A subsectors as the minimal difference between the highest energy CFT state and lowest generic state. The inset in (b) shows the finite-size scaling of the entanglement gap for the Coulomb state, which remains finite in the TD limit.

CL has the desirable property that the spacing between entanglement eigenvalues at the same angular momentum goes to zero very quickly as the sphere is enlarged, thus cementing the relation between entanglement energies and edge mode energies. The low-lying levels start by showing the universal CFT counting but then exhibit finite-size effects. For generic FQH states, obtained by diagonalizing the Coulomb Hamiltonian, the entanglement spectrum in the CL exhibits a *full* gap between *all* the model levels and the generic, high-energy Coulomb ones. This shows that not only the CFT-like levels are important in the determination of a state: the levels which exhibit finite-size effects are also a fingerprint of the state.

Diagonalizing a many-body Hamiltonian invariably introduces normalization factors of the noninteracting many-body states which depend on the specific geometry of the underlying manifold. In particular, these factors contain the information about the extent of the Landau orbitals in space, and depend on the magnetic length of the problem. Stated differently, this type of normalization relies on the curvature, i.e., a local quantity of the manifold. By contrast, the CL should by definition contain no real length scale. We are led to the conclusion that the best way to analyze a FQH polynomial obtained from the diagonalization of any Hamiltonian is to *unnormalize* it and strip it down to its magnetic length information. We now exemplify this procedure for the sphere geometry. Free boson states are spanned by the monomials $m_\lambda = 1/\sqrt{\prod_{j=0}^{\lambda_1} n_j!}\tilde{m}_\lambda$, where i runs over the number of particles N and j over the number of orbitals, and n_j denotes the multiplicity of occupation of the jth orbital. λ defines a partition of the angular momentum λ_j of different occupied orbitals, and Per denotes the permanent state with single particle positions z_i. The m_λ are free many-particle states that are *unnormalized*. When one diagonalizes a many-body Hamiltonian, the expansion of the interacting wave function is in *normalized* free many-body states $\mathcal{M}_\lambda$, which differ from the unnormalized basis above through normalization factors that contain information about the

geometry of the manifold and the magnetic length. On the sphere of radius R, the normalization of m_λ is given by [8]

$$\mathcal{N}_\lambda{}^{\text{sphere}} = \left(\frac{4\pi}{(2S+1)!}\right)^N \frac{N!}{\prod_{j=0}^{\lambda_1} n_j!} \prod_{i=1}^N \lambda_i!(2S-\lambda_i)!, \quad (1)$$

where n_j is the multiplicity of the jth orbital in the decreasingly ordered partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$, where $\lambda_i \in [0, 2S]$ is the angular momentum of the Landau orbitals. We set the partition to be padded, such that, if the initial partition has l_λ number of elements nonzero, then all the rest $\lambda_{l_\lambda+1} \ldots \lambda_N = 0$. The number of orbitals is conventionally given by $2S + 1$, where $N_\phi = 2S$ is the magnetic flux. We then apply the transformation

$$\mathcal{M}_\lambda = m_\lambda / \sqrt{\mathcal{N}_\lambda{}^{\text{sphere}}} \quad (2)$$

to write the state as a function of the *unnormalized* free boson many-body states. As an example, in the new basis, the *unnormalized* Laughlin state for two particles reads $m_{(2,0)} - 2m_{(1,1)}$. In the unnormalized basis, all the coefficients of the Laughlin state are integers. It is the basis which shows significant structural information about the polynomial [9,10]. For bosons, the last step is to normalize each of the new free many-body states by the square-root of the product of the factorials of their bosonic multiplicities $m_\lambda = \frac{1}{\sqrt{\prod_{j=0}^{\lambda_1} n_j!}}\tilde{m}_\lambda$. Once expressed in this new basis (the conformal limit), we calculate the ES for the ground states of different Hamiltonians.

As the first example, Fig. 1 illustrates the conformal limit transformation of the bosonic $\nu = 1/2$ Coulomb state. The sphere is partitioned into two parts A and B in the orbital space which mimics the geometrical partition [4,11]. The region A is made of the l_A first orbitals, starting from the north pole. In region A, the total number of particles N_A and the projection of total angular momentum L_z^A are good quantum numbers to define the different sectors of the ES. We define the ES to be the minimal difference between the highest energy CFT state and the
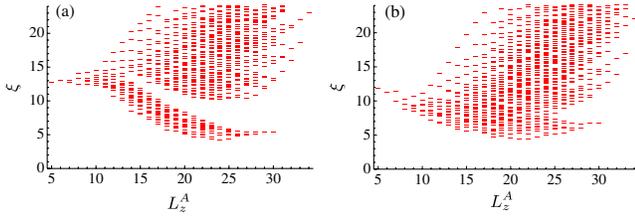
FIG. 2 (color online).   Entanglement spectrum at filling $\nu = 1/2$ for the $N = 11$ bosons and the ground state of Coulomb interaction with a modified short-range component by some δV_0 pseudopotential. Here, $N_\phi = 20$, $N_A = 5$, and $l_A = 10$. Left panel (a) is obtained for $\delta V_0 = -0.35$ where the gap starts closing. Right panel (b) is for $\delta V_0 = -0.425$, close to transition to an compressible $L \neq 0$ state.

lowest energy generic state of all different sectors. As shown in Fig. 1, the spectrum cleanly rearranges in a low-entanglement energy part and a high entanglement energy part, separated from each other by a homogeneous entanglement gap, unlike in the case of the sphere geometry where the entanglement gap can be defined for only a few L_z^A values. Moreover, the state counting in the low-energy part of the Coulomb spectrum exactly matches with the pure Laughlin spectrum for each L_z^A sector. In particular, this separation into a Lauglin-like low-energy spectrum and a higher-energy Coulomb continuum only holds for the CL basis and, independent of finite size or for the TD limit, is never reached for the sphere-normalized state.

We investigated the behavior of the gap when going through a phase transition toward a compressible state. A previous study for the ES in the sphere normalization has been done for $\nu = 1/3$ [12]. In a similar way, using the Haldane pseudopotentials decomposition of the Coulomb interaction, we modify the pseudopotential associated to the shortrange component by some amount δV_0 to drive the system into a compressible state. Starting from the Coulomb interaction at $\nu = 1/2$ for $N = 11$ bosons, the transition occurs at $\delta V_0 \simeq -0.45$. Figure 2 shows two particular values where the gap starts closing ($\delta V_0 \simeq -0.35$) and close to the transition point ($\delta V_0 \simeq -0.425$). The (square) overlap with the Laughlin state stays rather high (resp. 0.9895 and 0.9288). With such overlaps, one

would conclude that we are still in the same quantum phase. Here, the ES gives a more precise insight and tends to show that the transition may occur for larger δV_0. Still, there is no proof that as soon as the gap closes in one L_z^A sector, all topological properties are lost.

Our CL basis enables us to study whether different states are entanglement adiabatically continuable to each other. We conjecture that two states are entanglement adiabatically continuable if we can find a path to go from one state to the other without collapsing the full entanglement gap. If so, we conjecture that the states have identical topological structure. Let us illustrate this property with the example of a one parameter Hamiltonian that linearly interpolates between the three-body hard-core interaction for which the MR state [13] is the exact zero energy state [14], and the two-body hard-core interaction

$$\mathcal{H}_\lambda = (1-\lambda)\sum_{i<j<k}\delta(r_i-r_j)\delta(r_j-r_k) + \lambda\sum_{i<j}\delta(r_i-r_j).$$

(3)

Figure 3 shows spectra for several values of λ. We find that the spectra of the pure three-body hard-core potential and the two-body hard-core Hamiltonian are entanglement adiabatically connected within the ES. Finite-size scaling for the individual Hamiltonians also shows that the entanglement gap, though smaller for the Pfaffian case at $\nu = 1$ than for the previously studied Laughlin at $\nu = 1/2$, persists in the TD limit. Even though the overlaps between the ground state at $\lambda = 1$ and the MR state are lower (0.8858 for $N = 14$) than the ones we have previously mentioned in the $\nu = 1/2$ case close to the phase transition, in this case, there is a clear entanglement gap. This example clearly shows that high overlap is not a good indicator of a possible entanglement gap.

For the case of the fermionic wave functions of a generic Hamiltonian, one has to perform the same operations (2), while the occupation multiplicity terms are trivial. In Fig. 4, we show the rearrangement of the ES of the fermionic $\nu = 1/3$ Laughlin state upon the CL basis transformation. Notably, the universal CFT level part associated with the pure Laughlin state levels completely separates from the generic levels in the CL.
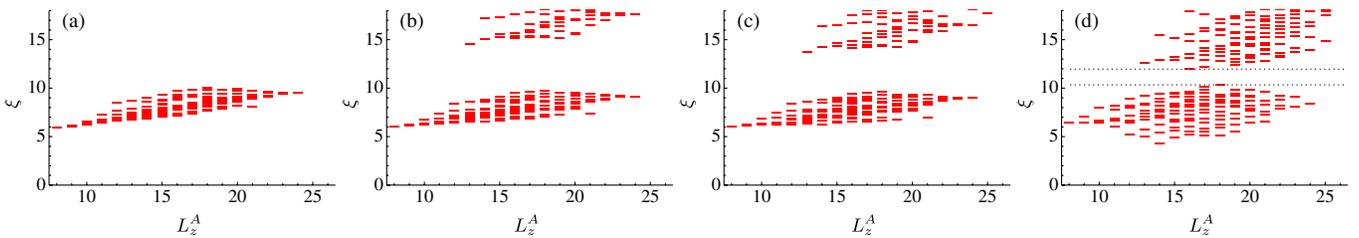
FIG. 3 (color online).   Entanglement spectra for the $\nu = 1$ ground state of (3) for $\lambda = 0$ [(a), pure MR state], 0.15 (b), 0.5 (c), and 1 [(d), pure delta ground state] for $N = 14$ bosons, $N_\phi = 12$, and the cut specified by $N_A = 7$, $l_A = 6$. We observe that the entanglement gap shrinks, but retains a finite value up to the pure two-body potential. The dashed lines refer to sector $L_z^A = 17$ where the minimum difference between the highest CFT level and lowest generic level, i.e., the entanglement gap, is found. The low-energy spectrum retains the identical structure and is adiabatically connected.
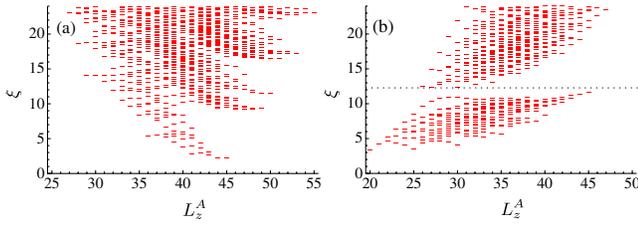
FIG. 4 (color online). Entanglement spectrum for $N = 11$ fermions Coulomb state at $\nu = 1/3$ filling, $N_\phi = 30$, and $N_A = 5$, $l_A = 14$. As in the bosonic case shown in Fig. 1, starting from the sphere normalization in (a), the CL basis in (b) clearly separates the universal states from the generic states in the spectrum indicated by the dashed line, and a clear entanglement gap over all L_z^A subsectors emerges.

As first shown in [4], the counting of low-energy entanglement levels can be related to the CFT edge theory of the state, which allows us to identify topological bulk properties. We here go further and investigate whether there is direct correspondence not only between the counting of the levels but also between the actual energies of the edge states and topological "entanglement energy" levels. From field theory [15], edge states obey a relativistic dispersion. If an entanglement level at $L_z^A = L_{z,\mathrm{max}}^A - m$ is identified to be related to an edge state level, it obeys $E = \sum_i v(2\pi/L)k_i$, where k_i is the momentum of the individual field, L the system length, v the velocity scale of the respective edge branch, and it holds $\sum_i k_i = m$. This implies that within one certain sector of total momentum m, which in terms of entanglement levels would correspond to the sector $L_{z,\mathrm{max}}^A - m$, all entanglement levels corresponding to different partitions of momentum m on different edge fields should have the same energy. Thus, in the TD limit where all finite-size effects are absent, we conjecture that the spread of the universal low-energy entanglement states in each L_z sector should shrink to zero, and the overall slope from one momentum sector to the other obeys a linear dispersion relation. We illustrate this for the $\nu = 1/2$ Laughlin state, for which the edge spectrum consists of one single bosonic branch and where we can go to suitably large system sizes (Fig. 5). We pick the highest L_z^A sectors $L_{z,\mathrm{max}}^A - m$ up to $m = 3$.

We first obtain the mean value of the low-energy states in one sector for the TD limit, and then extrapolate the dispersion relation with respect to m. We find that the extrapolated dispersion is linear within moderate error, confirming the relativistic behavior of these entanglement levels. We also find that the spread of the low-energy levels shrinks to zero in the TD limit. While this holds in geometry, the CL basis makes this feature become apparent already for small system sizes, as the low-energy levels in one sector become significantly squeezed (Fig. 5). Our results thus confirm that the low-energy ES modes correspond qualitatively and quantitatively to the real edge modes existing in the FQH state. This fortifies the statement that if the low-entanglement structure of two states is
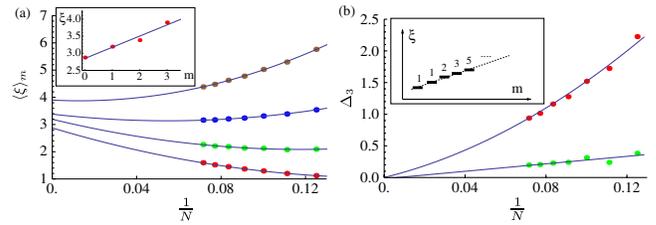
FIG. 5 (color online). (a) Fit of mean values $\langle \xi \rangle_m$ in the $\nu = 1/2$ Laughlin state for different sectors $L_{z,\mathrm{max}}^A - m$, for $m = 0, 1, 2, 3$. As system size increases, the individual entanglement levels go up in "entanglement energy." The best fit is accomplished by an inverse quadratic fit to the number of bosons. The inset shows the extrapolated linear dispersion relation for the edge states. (b) Scaling of the spread Δ_3 of the $L_{z,\mathrm{max}}^A - 3$ sector for the sphere-normalized Laughlin state approximated by a superlinear fit and the Laughlin state in the CL by a linear fit. Both extrapolate to zero width in the TD limit. The inset shows the expected shape of the Laughlin ES spectrum for infinite system size—a linear dispersive set of states following the edge mode state counting and degenerate in the different L_z^A sectors.

similar, reflecting stable edge modes which characterize a topological state, then two states with the same entanglement structure are identified to be in the same topological universality class.

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