One-Parameter Scaling Theory for Stationary States of Disordered Nonlinear Systems

Joshua D. Bodyfelt,[1,2,3] Tsampikos Kottos,[1] and Boris Shapiro[4]

[1]*Department of Physics, Wesleyan University, Middletown, Connecticut 06459, USA*
[2]*MPI for Dynamics and Self-Organization, Bunsenstraße 10, 37073 Göttingen, Germany*
[3]*MPI for the Physics of Complex Systems, Nöthnitzer Straße 38, D-01187 Dresden, Germany*
[4]*Technion-Israel Institute of Technology, Technion City, Haifa 32000, Israel*
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We show, using detailed numerical analysis and theoretical arguments, that the normalized participation number of the stationary solutions of disordered nonlinear lattices obeys a one-parameter scaling law. Our approach opens a new way to investigate the interplay of Anderson localization and nonlinearity based on the powerful ideas of scaling theory.

Introduction.—Wave propagation in naturally occurring or engineered complex media is an interdisciplinary field of research that addresses systems as diverse as classical, quantum, and atomic-matter waves. Despite this diversity, the wave nature of these systems provides a common framework for understanding their transport properties and often leads to new applications. One such characteristic is wave interference phenomena. Their existence results in a complete halt of wave propagation in random media, which can be achieved by increasing the randomness of the medium. This phenomenon was predicted 50 years ago in the framework of quantum (electronic) waves by Anderson [1] and its existence has been confirmed in recent years in experiments with classical [2–10] and matter waves [11,12].

In many of these experiments, the appearance of nonlinearities, induced either due to the nonlinear Kerr effect (in the framework of nonlinear wave propagation in disordered photonic lattices) [8–10] or due to atom-atom interactions (in atomic transport of Bose-Einstein condensates in optical lattices) [11,12], might affect drastically the phenomenon of Anderson localization. An important question is therefore how the interplay between disorder and nonlinearity might complement, frustrate, or reinforce each other [13–16]. This notion is not only of experimental importance; it raises a number of unsolved theoretical questions as well. The theoretical study of localization in random nonlinear lattices has been advanced using several approaches, including the studies of transmission [17], wave packet dynamics [18], and stationary solutions [19–21].

In this Letter, we approach the interplay of nonlinearity with disorder from a different perspective; namely, we develop a scaling theory for localization phenomena in nonlinear random media described by the discrete nonlinear Schrödinger equation (DNLSE). Scaling ideas played a major role in understanding various properties of *linear* disordered systems, including the structure of their eigenstates [22–25]. However, solutions of the DNLSE, for sufficiently strong nonlinearity, have nothing in common with the solutions of the linear problem. Indeed, the number of solutions of the DNLSE is generally much larger than the number of eigenstates of the corresponding linear problem, and in particular, many solutions appear outside the spectrum of the linear system. It is therefore quite remarkable that the scaling ideas can be extended to the nonlinear case. Specifically, we find that the rescaled participation number $p_N(\chi)$ of the stationary solutions of the DNLSE of lattice size N and nonlinearity χ obeys a one-parameter scaling, i.e.,

$$\frac{\partial p_N(\chi)}{\partial \ln N} = \beta(p_N(\chi)) \quad \text{where} \quad p_N(\chi) = \frac{\langle \xi_N(\chi) \rangle}{\langle \xi_N^{\text{ref}}(\chi) \rangle}. \quad (1)$$

Above, β is a *universal* function of p_N alone, which is independent of any microscopic properties of the system under investigation, and $\langle \cdots \rangle$ denotes an averaging over disorder realizations and over states within a small frequency window. The participation number ξ_N is [26]

$$\xi_N \equiv \frac{1}{\sum_{n=1}^{N} |\psi_n|^4} \quad (2)$$

and is proportional to the effective number of nonzero components ψ_n of a stationary solution of the DNLSE. ξ_N^{ref} is the participation number for some reference ensemble, chosen such that it supports the most extended states for a specific lattice topology and nonlinearity. It will be argued below that ξ_N^{ref} is proportional to the system size N. Note, however, that it would not be accurate to replace ξ_N^{ref} with N because the coefficient of proportionality between the two lengths depends weakly on χ. Equation (1) is confirmed in the following via detailed numerical simulations with quasi-one-dimensional disordered systems described by banded random matrix (BRM) models and is supported by theoretical arguments [27].

Mathematical model.—We consider a class of *random* systems described by the time-independent DNLSE:

$$\sum_m H_{nm}\psi_m + \chi|\psi_n|^2\psi_n = \omega\psi_n, \quad (3)$$

where ω are the frequencies of the stationary solutions and ψ_n is their amplitude at site n. The connectivity matrix H

defines the topology of the sample. In the case of strictly one-dimensional (1D) disordered systems, it is a three-diagonal matrix with $H_{n,n\pm1} = -1$, while H_{nn} are random independent variables given by some distribution. In our simulations below, we consider the challenging case where H belongs to a BRM ensemble, having in mind quasi-1D systems with b propagating modes [23,24].

The BRM ensemble is defined as a set of real symmetric $N \times N$ matrices with elements $H_{nm} = 0$ for $|n - m| \geq b$, while inside the bandwidth b they are independent random variables given by a Gaussian distribution of mean zero and fixed variance [23]

$$\langle H_{nm}\rangle = 0, \qquad \langle H_{nm}^2\rangle = \frac{(N+1)(1+\delta_{nm})}{b(2N-b+1)}. \qquad (4)$$

With this normalization, the eigenvalues of H (in the limit of large N, b) are located in the interval $(-2, 2)$ [23].

Numerical method.—The stationary solutions of the nonlinear Eq. (3) were found numerically by utilizing a continuation method approach. Starting with the linear modes of H as an initial guess, we take a small step in χ ($\delta\chi \sim 10^{-4}$), and solve the nonlinear system of Eq. (3). This is achieved by minimizing a multivariable $(N + 1)$-dimensional vector function $\vec{F}_n = \sum_m H_{nm}\psi_m + \chi|\psi_n|^2\psi_n - \omega\psi_n$, $n = 1, 2, \ldots, N$ and $\vec{F}_{N+1} = \sum_n|\psi_n|^2 - 1$. Using this method we find $(\{\psi_n\}; \omega)$ with tolerance 10^{-8}. The resulting solution then becomes the initial guess for the nonlinear solver for the next step in χ.

An "evolution" of a representative Anderson localized mode as nonlinearity χ increases is reported in Fig. 1. We observe that while initially (small χ values) its shape remains unaffected, eventually it starts to delocalize until it spreads over the whole sample. The degree of delocalization as a function of χ is reflected in the behavior of the participation number $\xi_N(\chi)$ (lower panel of Fig. 1). We want to investigate how the average participation number $\langle\xi_N\rangle$ of the stationary modes of Eq. (3) is affected by nonlinearity. The averaging $\langle\cdots\rangle$ has been performed over solutions with corresponding ω being inside a small frequency interval (below $\omega \in [-1, 1]$) such that the nature of wave functions is statistically the same. For better statistical processing, a number of disorder realizations have been used, such that the total number of obtained stationary solutions is at least 10^4.

Nonlinear localization length.—We start our analysis by introducing the asymptotic localization length λ_χ defined via the participation number ξ_N:

$$\lambda_\chi = \lim_{N\to\infty}\langle\xi_N(\chi)\rangle. \qquad (5)$$

In Fig. 2(a) we report some representative data for the participation number $\langle\xi_N(\chi)\rangle$, as a function of the system size N. From these plots we extract the saturation value λ_χ. The resulting data are summarized in Fig. 2(b) by referring to the rescaled localization length λ_χ/λ_0, where λ_0 is the localization length given by Eq. (5) for the linear (i.e., $\chi = 0$) system [28]. We find that
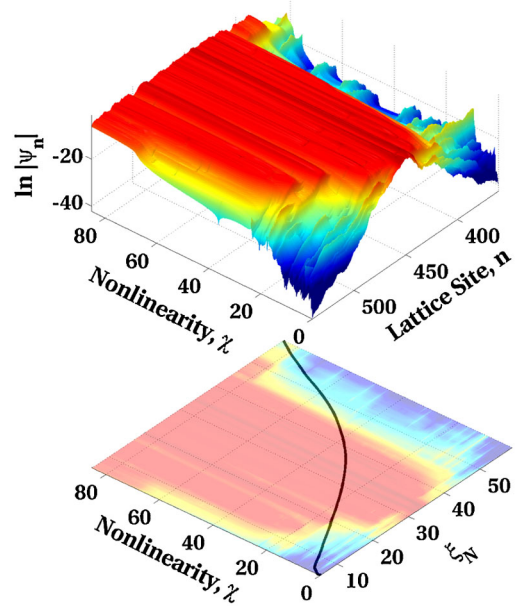

FIG. 1 (color online).   Upper panel: Parametric evolution as the nonlinearity χ increases of a stationary solution of the BRM, Eq. (3), with $b = 4$. For $\chi = 0$ the mode is exponentially localized. while as χ increases it becomes delocalized over the whole lattice. Lower panel: The participation number $\xi_N(\chi)$ (solid black line) as a function of χ. The coloring reflects the wave function intensity shown in the upper panel.

$$\frac{\lambda_\chi}{\lambda_0} \propto \begin{cases} 1 & \text{for } \chi < \chi^* \\ \sqrt{\chi} & \text{for } \chi \gg \chi^*, \end{cases} \qquad (6)$$

where $\chi^* \sim 0.3$ [18]. A simple interpolation formula which agrees with our numerical results [see Fig. 2(b)] is

$$\lambda_\chi = \lambda_0\sqrt{1 + a_0\chi}, \qquad (7)$$

where the fitting parameter a_0 was found to be $a_0 \approx 3$. While interpreting the above equations, it is crucial to keep in mind that all the lengths depend on frequency ω—when comparing these lengths for different values of the nonlinearity χ, one should keep ω (approximately) fixed.

The following heuristic argument provides some understanding of the behavior of λ_χ in the two limiting cases. First, let us note that the inverse of the participation number $\xi_N^{-1}(\chi)$ is proportional to the interaction energy stored in the system described by the DNLSE, i.e., $E_{\text{int}} = \frac{\chi}{2}\sum_n|\psi_n|^4$. For $\chi = 0$ the localization length λ_χ is equal to λ_0, while due to normalization the corresponding stationary solutions of Eq. (3) are $\psi_n^{(0)} \sim 1/\sqrt{\lambda_0}$. When χ increases, E_{int} grows within the localization length λ_0, but ω is assumed to be approximately fixed. This increase in E_{int} is compensated by further spreading of the wave function beyond λ_0. The first question is, what is the value of the nonlinearity strength χ^*, for which the spreading beyond λ_0 becomes significant? An estimation is achieved by comparing the interaction energy $E_{\text{int}} = (\chi/2)\sum_n|\psi_n^{(0)}|^4 \approx (\chi/2)\lambda_0(1/\lambda_0^2) = \chi/2\lambda_0$ stored in the
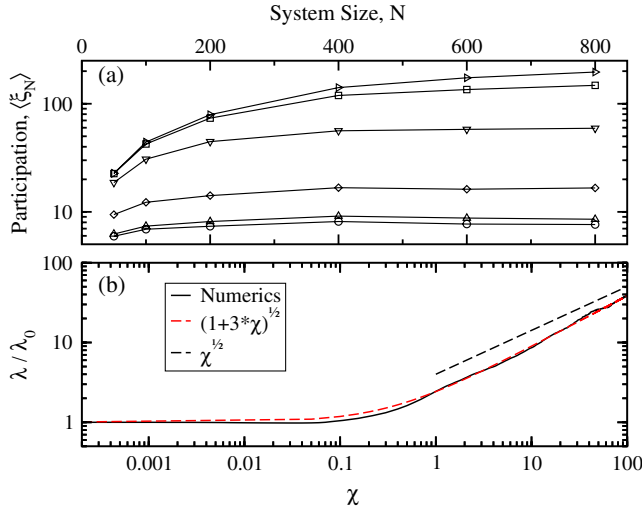
FIG. 2 (color online).   (a) The participation number $\xi_N(\chi)$ vs the system size N for various nonlinearity strengths: $\chi = 0$ ($\bigcirc$), $\chi = 0.1$ ($\triangle$), $\chi = 1$ ($\diamond$), $\chi = 10$ ($\triangledown$), $\chi = 50$ ($\square$), $\chi = 100$ ($\triangleright$). The asymptotic localization length λ_χ is evaluated by a direct fit of the saturation value of $\xi_N(\chi)$ in the limit $N \to \infty$. (b) The extracted λ_χ versus the nonlinearity strength χ. The black dashed line has slope 0.5 and is drawn in order to guide the eye. The lighter (red) dashed line is the best fitting curve Eq. (7).

"localization box" of size λ_0 to the corresponding mean level spacing $\Delta_{\lambda_0} \sim 1/\lambda_0$. From this comparison, we get $\chi^* \sim 1$.

Next, we consider the limit $\chi \gg \chi^*$, when $\lambda_\chi \gg \lambda_0$. In this case, the wave functions $\psi^{(\chi)} \sim 1\sqrt{\lambda}$ spread over λ_χ/λ_0 "localization boxes." We make the following self-consistent argument: The total interaction energy stored is $E_{\text{int}} = (\chi/2)\sum_n |\psi_n^{(\chi)}|^4 \approx (\chi/2)\lambda_\chi(1/\lambda_\chi^2) = \chi/2\lambda_\chi$. The interaction energy per box is therefore $(\chi/2\lambda_\chi)(\lambda_0/\lambda_\chi) = (\chi/2)\lambda_0/\lambda_\chi^2$. However, from the previous considerations we know that one localization box can "resist" an energy $\Delta_{\lambda_0} \sim 1/\lambda_0$. A self-consistency condition gives $\lambda_\chi \sim \lambda_0\sqrt{\chi}$, which agrees with the $\chi \gg \chi^*$ asymptotic in Eq. (6).

Reference ensemble.—The most ergodic stationary solutions of Eq. (3) correspond to connectivity matrices H, taken from the Gaussian orthogonal ensemble [29]. These solutions will spread over the entire system, of size N. However, this does not mean $\xi_N^{\text{ref}} = N$, since $\psi_n^{(\chi)}$ might have some (oscillatory) structure. We therefore assume that

$$\xi_N^{\text{ref}}(\chi) = N\alpha(\chi) \quad \text{where } 1/3 \le \alpha(\chi) \le 1. \quad (8)$$

The lower value of $\alpha(\chi)$ is achieved in the limit of $\chi \to 0$, where $\langle \xi_N^{\text{ref}} \rangle = N/3$ [23,29]. The fact that $\langle \xi_N^{\text{ref}} \rangle$ is 3 times less than the system size is due to Gaussian fluctuations in the components ψ_n. The other limiting case of $\chi \to \infty$ corresponds to $\alpha = 1$. Indeed, strong nonlinearity favors a completely uniform $|\psi_n|^2$ (again, for fixed and moderate ω). The following argument, similar in spirit to the "maximum entropy" ansatz, can provide some understanding.

Let us denote the components $\psi_n^{(\chi)}$ of a stationary solution by a random variable x. Assume that the variable x follows a distribution $\mathcal{P}(x)$, "as random as possible" but with the constraint $\mathcal{N} = \int dx x^2 \mathcal{P}(x) = 1/N$ dictated by normalization of the wave function. For $\chi = 0$, it was shown that the "most random" distribution is the Gaussian. It can be found by maximizing the entropy $S = -\int dx \mathcal{P}(x) \ln \mathcal{P}(x)$ with the above constraint [30]. For $\chi \ne 0$, one however has also to consider the increase of interaction energy E_{int}. The entropy favors a distribution $\mathcal{P}(x)$, "as random as possible" for the values of x at various sites; e.g., the distribution of the "weight" when all of the probability is on a single site is as likely as the configuration with equal values of all the sites. The energy however favors a uniform distribution—clearly, a "single site weight" will require huge energy. Thus, the correct quantity to minimize is the "free energy" functional $F[\mathcal{P}] = -S + E_{\text{int}} + \beta\mathcal{N}$. We get

$$\mathcal{P}(x) = C \exp[-\beta x^2 - (1/2)\chi x^4], \quad (9)$$

where C and β are defined from the constraints $\mathcal{N} = 1/N$ and $\int \mathcal{P}(x)dx = 1$. For $\chi = 0$ the Gaussian distribution is recovered, whereas for $\chi \to \infty$ the distribution turns δ-like around $x = 1/\sqrt{N}$.

One-parameter scaling ansatz.—We are now equipped to formulate a scaling theory for the stationary solutions of Eq. (3). The scaling ansatz of Eq. (1) is equivalent to postulating the existence of a function $f(x)$ such that [31]

$$\frac{\langle \xi_N(\chi) \rangle}{\langle \xi_N^{\text{ref}}(\chi) \rangle} = f(x) \quad \text{where } x = \frac{\lambda_\chi}{\langle \xi_N^{\text{ref}}(\chi) \rangle}. \quad (10)$$

In the delocalized limit $x \gg 1$, the function $f(x)$ approaches the saturation value 1. On the other hand, when $N \to \infty$ (i.e., $x \ll 1$) we have that $\langle \xi_N(\chi) \rangle \to \lambda_\chi$; thus $f(x) = x$. We have numerically tested Eq. (10) for the DNLSE using a BRM ensemble for the connectivity matrix H. Various values of N and b in the ranges $50 \le N \le 800$ and $1 \le b \le N/2$ were used in the analysis. The numerical data are reported in Fig. 3, confirming nicely the scaling ansatz of Eq. (10). Finally, it is reassuring that Eq. (10) in the limit $\chi = 0$ recovers the scaling relation found for linear disordered lattices [23–25].

It is clear that "strong" discrete breathers localized at few sites are excluded from our above considerations. In fact, such solutions correspond to large frequency values $\omega \sim \chi$, which for very large χ are always outside *any* fixed small frequency window over which our scaling analysis is performed. The analysis of discrete breathers can be done separately, since they originate from the clean system with large χ and ω, in which disorder has only a minor effect. Thus, all lengths appearing in Eq. (10) are (by their definition) approximately the same, i.e., $\xi_N^{\text{ref}} \sim \xi_N(\chi) \sim \lambda_\chi$, on the order of the lattice spacing. This is a rather trivial limit, where Eq. (10) is again expected to be approximately valid.
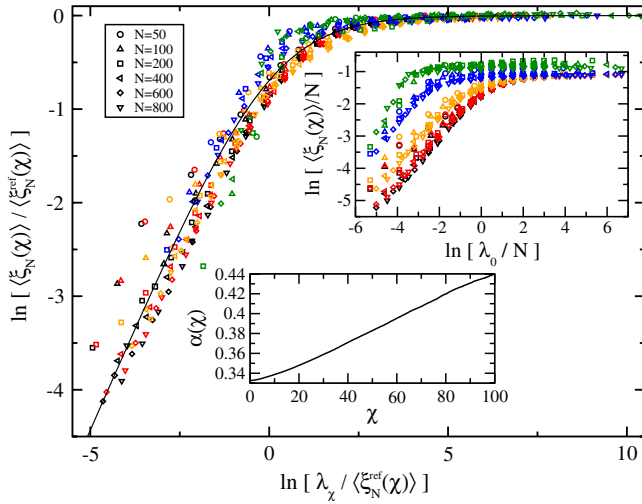
FIG. 3 (color online). Main panel: Participation ratio $\xi_N(\chi)/\xi_N^{\mathrm{ref}}(\chi)$ for the BRM model vs the rescaled parameter $\lambda_\chi/\xi_N^{\mathrm{ref}}(\chi)$ for various (N, b) values (for each N we have used at least 12 different b values in the interval $1 \leq b \leq N/2$). Colors correspond to nonlinear values of $\chi = 0$ (black), $\chi = 0.1$ (red), $\chi = 1$ (orange), $\chi = 10$ (blue), and $\chi = 100$ (green). The solid line is the best fit with the function $y = 0.98x/(1 + 0.98x)$. Upper inset: The same data as in the main panel, reported using the scaling variables $\langle \xi_N \rangle/N$ and λ_0/N. Lower Inset: The numerically evaluated $\alpha(\chi)$ vs χ for the nonlinear Gaussian orthogonal ensemble reference ensemble. The size of the matrix H is $N = 800$. In the limit $N \to \infty$ and for $\chi \gg 1$, the function $\alpha(\chi) \to 1$.

Conclusions.—We presented a one-parameter scaling theory for the average participation number of the stationary solutions of low-dimensional disordered nonlinear systems described by the DNLSE. Via numerical analysis and theoretical considerations, we have established Eq. (1) which allows us to conclude that changing disorder strength, nonlinearity, and frequency, in the way described by Eq. (10), would not change the (average) spatial extent of the stationary states of the DNLSE. The one-parameter scaling theory presented here is a powerful approach in the quest of understanding the interplay between nonlinearity and disorder. Although the focus of this Letter was on the structure of stationary modes, our approach can further be used to understand various other observables [32].

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[27] However, all our results can be immediately translated to the case of strictly 1D disordered systems.
[28] For $\chi = 0$, the localization length λ_0 is given by the inverse of the asymptotic decay rate of the wave functions (measured by the so-called Lyapunov exponent).
[29] For strictly 1D tight-binding models the reference system has a three-diagonal connectivity matrix H, with $H_{nn} = 0$. In this case, for $\chi = 0$ we get $\xi_N^{\mathrm{ref}} = 2N/3$ [25].
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[31] Note that Eq. (10) can be rewritten in the form $\langle \xi_N(\chi) \rangle / \lambda_\chi = F(x)$ where $F(x) = f(x)/x$.
[32] J. Bodyfelt, T. Kottos, and B. Shapiro (to be published).