

Bénard–von Kármán Vortex Street in a Bose-Einstein Condensate

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Vortex shedding from an obstacle potential moving in a Bose-Einstein condensate is investigated. Long-lived alternately aligned vortex pairs are found to form in the wake, which is similar to the Bénard–von Kármán vortex street in classical viscous fluids. Various patterns of vortex shedding are systematically studied and the drag force on the obstacle is calculated. It is shown that the phenomenon can be observed in a trapped system.

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The formation of a train of alternate vortices in the wake past an obstacle, known as the Bénard–von Kármán vortex street, is a ubiquitous and intriguing phenomenon in fluids. Since the pioneering experimental study by Bénard [1] and theoretical consideration by von Kármán [2], numerous studies have been made on the phenomena of vortex street formation [3].

The behavior of a viscous fluid flowing past an obstacle is determined by the Reynolds number Re , which is a dimensionless parameter that includes the kinematic viscosity. The wake of a cylinder is steady for Re below around 50, and a vortex street emerges for $10^2 \leq Re \leq 10^5$, which becomes turbulence for a larger Re . This implies that the viscosity plays an important role in vortex street formation in classical fluids. For superfluids, however, the Reynolds number cannot be defined because of the absence of viscosity. Moreover, the vortex quantization makes superfluid dynamics quite different from classical fluid dynamics. Therefore, it is not obvious whether the instability of the wake and subsequent vortex street generation occur in superfluids. According to von Kármán's theory [2,4], a vortex street is expected to be very long-lived in inviscid fluids, once it is created.

In this Letter, by numerically solving the Gross-Pitaevskii (GP) equation, we show that long-lived alternately aligned vortex pairs are formed in the wake of an obstacle potential moving in a Bose-Einstein condensate (BEC), which is similar to the vortex street in classical fluids. Mean-field analysis of systems of a BEC with a moving potential has been performed by many authors from the viewpoints of drag force [5–7], vortex dynamics near the cylinder [8,9], critical velocity [10], scaling laws [11], supersonic flows [12,13], and multicomponent systems [14,15]. However, vortex street formation was not found in these studies, probably because the parameter region for which a vortex street emerges is narrow, as shown later.

We consider a BEC of atoms with mass m and an obstacle potential V moving in the $-x$ direction at a velocity v . In the mean-field theory, the condensate is described by the macroscopic wave function ψ obeying the GP equation given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi + g|\psi|^2\psi, \quad (1)$$

where $g = 4\pi\hbar^2 a/m$ with a being the s -wave scattering length of the atoms. We employ a Gaussian potential with peak strength V_0 and radius d moving in the $-x$ direction at a velocity v as $V = V_0 \exp\{-[(x+vt)^2 + y^2]/d^2\}$. Normalizing space and time by $\hbar/(mgn_0)^{1/2}$ and $\hbar/(gn_0)$, where n_0 is the atom density without perturbation, we can eliminate the interaction parameter g from Eq. (1). We numerically solve Eq. (1) in two dimensions under the periodic boundary condition using the pseudospectral method. The initial state is the stationary state of Eq. (1) with $v = 0$ plus a small amount of noise to break the symmetry.

Figure 1 shows typical wakes flowing past an obstacle potential with $V_0/(gn_0) = 100$. For a sufficiently small velocity v , the flow around the obstacle is a steady laminar flow and no quantized vortex is created. When the velocity v exceeds a critical velocity, which depends on the interaction strength and the shape of the potential, vortex-antivortex pairs are created [5]. The critical velocity for the vortex creation is of the order of the speed of sound $(gn_0/m)^{1/2}$. When created vortices separate from the potential, the flow velocity around the potential again exceeds the critical velocity and subsequent vortices are created. A train of vortex-antivortex pairs is thus generated behind the potential. Since a symmetric double row of vortices is unstable [4], the vortex pairs are dislocated sinusously as shown in Fig. 1(a). Nore *et al.* [16] showed that such a staggered vortex pattern is formed if a double row of vortices is prepared with an appropriate perturbation. Since a pair of point vortices with circulations $\pm h/m$ (h : Planck's constant) and distance d moves in the direction perpendicular to a line between the pair at a velocity $\hbar/(md)$ [4], the alternately inclined vortex pairs move in two directions [white arrows in Fig. 1(a)], forming a V-shaped wake as in Fig. 1(a). The divergence of the wake is significant for large v with small d , which forms a pattern similar to supersonic flow [12,13].

Figure 1(b) shows the main result of this study. The significant difference from Fig. 1(a) is that the vortices in

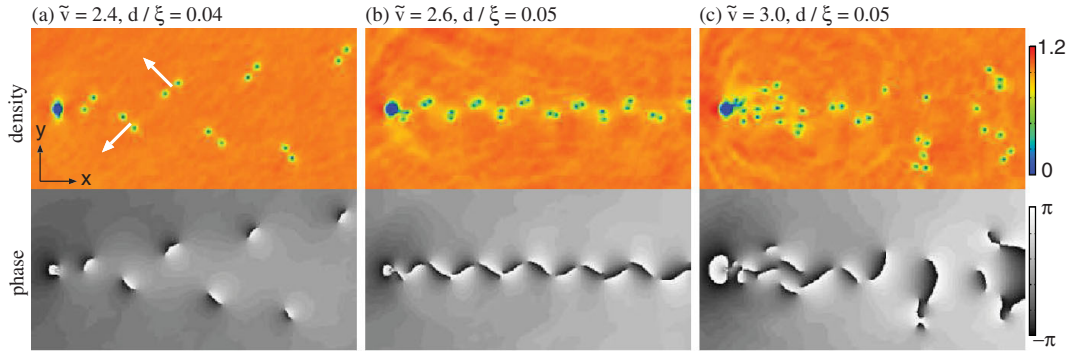


FIG. 1 (color). Density and phase distributions of a condensate past an obstacle potential. The velocity and potential width are $(\tilde{v}, d/\xi) = (2.4, 0.04)$ in (a), $(2.6, 0.05)$ in (b), and $(3.0, 0.05)$ in (c), where $\tilde{v} = v[10^3 m/(gn_0)]^{1/2}/(2\pi)$ and $\xi = \hbar[10^3/(mgn_0)]^{1/2}$. The white arrows in (a) indicate the directions in which the vortex-antivortex pairs move. The density is normalized by n_0 . The field of view is $6\xi \times 3\xi$. In the numerical calculation, a $32\xi \times 8\xi$ space is discretized into 4096×1024 .

a pair created by the obstacle potential at a time have the same circulation. Since two point vortices having the same circulation h/m rotate around their center at an angular frequency $2\hbar/(md^2)$ without changing their distance [4], the created vortex pairs in Fig. 1(b) remain bound and rotate. The pairs with opposite circulations are alternately released from the obstacle potential to form a train of vortex pairs, resembling a Bénard–von Kármán vortex street [17]. In contrast to the vortex arrangement originally considered by von Kármán, in which isolated point vortices are aligned, the vortex pairs constitute the vortex street in the present case. The detailed dynamics of the vortex street formation just behind the obstacle potential is shown in Fig. 2. The pairs of vortices are released obliquely

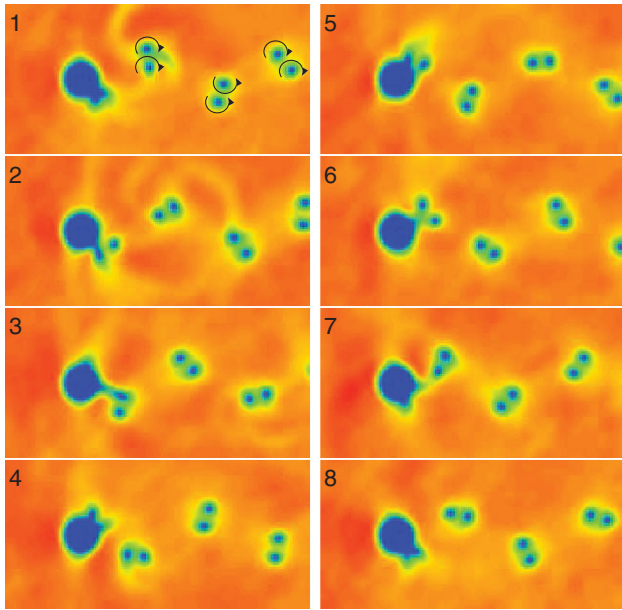


FIG. 2 (color). Serial snapshots of the density profiles for the parameters in Fig. 1(b) in the frame moving with the potential. The time interval is $10\hbar/(gn_0)$. The arrows indicate the directions of circulations. The field of view is $2\xi \times \xi$. The color scale is the same as that in Fig. 1.

backward left and right with alternate circulations. We find from Figs. 1(b) and 2 that the distance between the two vortex rows is $b \approx 0.24\xi$ and the distance between two pairs in a row is $\ell \approx 0.87\xi$ on average, and hence $b/\ell \approx 0.28$, where $\xi = \hbar[10^3/(mgn_0)]^{1/2}$. This ratio is in good agreement with the stability condition of von Kármán's vortex arrangement $b/\ell = \pi^{-1} \cosh^{-1} \sqrt{2} \approx 0.28$ [4]. In fact, the vortex street in Fig. 1(b) survives at least $t > 10^3 \hbar/(gn_0)$. The vortex street in Fig. 1(b) moves in the $-x$ direction at a velocity $\approx 0.14(gn_0/m)^{1/2} \approx 0.8h/(\sqrt{2}\ell m)$. The velocity of von Kármán's point vortices, in which each vortex has a circulation $2h/m$, is $h/(\sqrt{2}\ell m)$ [4]. For large d and v , the periodicity in the wake seems to disappear [Fig. 1(c)].

We systematically performed numerical simulations for various values of d and v to determine the parameter regions for the types of wakes in Fig. 1. Figure 3 shows a rough sketch of each parameter region. The regions of the

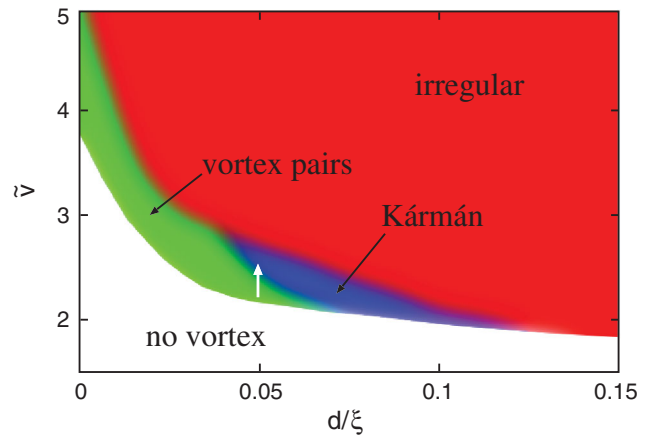


FIG. 3 (color). Dependence of the patterns of wakes on the normalized Gaussian width d/ξ and velocity \tilde{v} of the potential. The green, blue, and red regions correspond to the flow patterns shown in Figs. 1(a)–1(c), respectively. The white region corresponds to stationary laminar flow. The white arrow indicates the change of \tilde{v} used in Fig. 4.

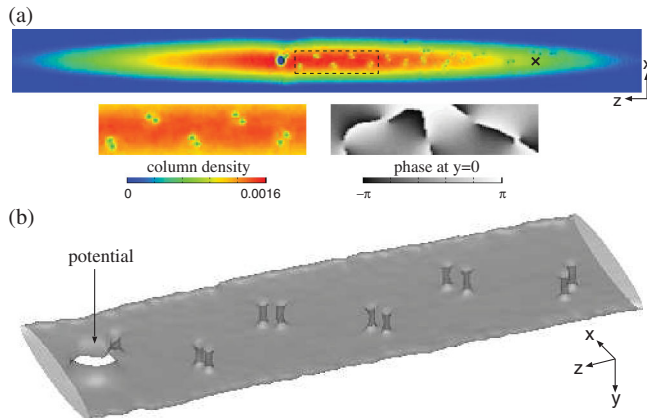


FIG. 6 (color). A BEC of ^{87}Rb atoms confined in a harmonic potential at $t = 160$ ms, where the Gaussian potential is initially located at the marked position (\times) and starts to move in the z direction at a constant velocity. (a) Column density normalized by $Nm\omega_x/\hbar$. The dashed region is magnified in the lower panels. The gray scale image shows the phase at $y = 0$. The field of view is $300 \times 30 \mu\text{m}$ for the main panel and $40 \times 10 \mu\text{m}$ for the lower panels. (b) Translucent display of the isodensity surface for the state shown in (a).

$r > d$) also generates vortex patterns similar to those in Fig. 1. This potential can be realized by a flattop laser beam or a Gaussian beam in which limit $V_0 \rightarrow \infty$ and $d \rightarrow 0$ is taken with $d^2 \log(V_0/\mu)$ being fixed, where μ is the chemical potential.

A future prospect of this study is to provide a mathematical description of the alternate creation of the vortex pairs as shown in Fig. 2. From an experimental point of view, it is important to study the cases of various trapping potentials including cigar-shaped and oblate traps. Various shapes of obstacle potentials also merit further study.

In conclusion, we have shown that pairs of quantized vortices shed from an obstacle potential moving in a BEC alternately align and survive for a long time [Figs. 1(b) and 2], which closely resembles the Bénard–von Kármán vortex street in classical fluids. We have obtained the parameter region in which a vortex street emerges (Fig. 3) and calculated the drag force (Fig. 5). We have shown that vortex street formation can be observed in a trapped BEC disturbed by a blue-detuned laser beam (Fig. 6). Since the vortex street generation typifies the great diversity of classical fluid dynamics, the emergence of the vortex street in a BEC implies that a rich variety of phenomena is still unrevealed in quantum hydrodynamics.

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