

Driving Sandpiles to Criticality and Beyond

Anne Fey,¹ Lionel Levine,² and David B. Wilson³

¹*Delft Institute of Applied Mathematics, Delft University of Technology, The Netherlands*

²*Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

³*Microsoft Research, Redmond, Washington 98052, USA*

(Received 16 December 2009; revised manuscript received 4 March 2010; published 9 April 2010)

A popular theory of self-organized criticality relates driven dissipative systems to systems with conservation. This theory predicts that the stationary density of the Abelian sandpile model equals the threshold density of the fixed-energy sandpile. We refute this prediction for a wide variety of underlying graphs, including the square grid. Driven dissipative sandpiles continue to evolve even after reaching criticality. This result casts doubt on the validity of using fixed-energy sandpiles to explore the critical behavior of the Abelian sandpile model at stationarity.

DOI: 10.1103/PhysRevLett.104.145703

PACS numbers: 64.60.av, 45.70.Cc

In a widely cited series of papers [1–5], Dickman, Muñoz, Vespignani, and Zapperi (DMVZ) developed a theory of self-organized criticality as a relationship between driven dissipative systems and systems with conservation. This theory predicts a specific relationship between the Abelian sandpile model of Bak, Tang, and Wiesenfeld [6], a driven system in which particles added at random dissipate across the boundary, and the corresponding “fixed-energy sandpile”, a closed system in which the total number of particles is conserved.

After defining these two models and explaining the conjectured relationship between them in the DMVZ paradigm of self-organized criticality, we present data from large-scale simulations which strongly indicate that this conjecture is false on the two-dimensional square lattice. We then examine the conjecture on some simpler families of graphs in which we can provably refute it.

Early experiments [7] already identified a discrepancy, at least in dimensions 4 and higher, but later work focused on dimension 2 and missed this discrepancy (it is very small). Some recent papers (e.g., [8]) restrict their study to stochastic sandpiles because deterministic sandpiles belong to a different universality class, but there remains a widespread belief in the DMVZ paradigm for both deterministic and stochastic sandpiles [9,10].

Despite our contrary findings, we believe that the central idea of the DMVZ paradigm is a good one: the dynamics of a driven dissipative system should in some way reflect the dynamics of the corresponding conservative system. Our results point to a somewhat different relationship than that posited in the DMVZ series of papers: the driven dissipative model exhibits a second-order phase transition at the threshold density of the conservative model.

Bak, Tang, and Wiesenfeld [6] introduced the Abelian sandpile as a model of self-organized criticality; for mathematical background; see [11]. The model begins with a collection of particles on the vertices of a finite graph. A vertex having at least as many particles as its degree topples by sending one particle along each incident edge.

A subset of the vertices are distinguished as sinks: they absorb particles but never topple. A single time step consists of adding one particle at a random site, and then performing topplings until each nonsink vertex has fewer particles than its degree. The order of topplings does not affect the outcome [12]. The set of topplings caused by the addition of a particle is called an avalanche.

Avalanches can be decomposed into a sequence of “waves” so that each site topples at most once during each wave. Over time, sandpiles evolve toward a stationary state in which the waves exhibit power-law statistics [13] (though the full avalanches seem to exhibit multifractal behavior [14,15]). Power-law behavior is a hallmark of criticality, and since the stationary state is reached apparently without tuning of a parameter, the model is said to be self-organized critical.

To explain how the sandpile model self-organizes to reach the critical state, Dickman *et al.* [1,3] introduced an argument which soon became widely accepted: see, for example, [[16], Ch. 15.4.5] and [17–19]. Despite the apparent lack of a free parameter, they argued, the dynamics implicitly involve the tuning of a parameter to a value where a phase transition takes place. The phase transition is between an active state, where topplings take place, and a quiescent “absorbing” state. The parameter is the *density*, the average number of particles per site. When the system is quiescent, addition of new particles increases the density. When the system is active, particles are lost to the sinks via toppling, decreasing the density. The dynamical rule “add a particle when all activity has died out” ensures that these two density changing mechanisms balance one another out, driving the system to the threshold of instability.

To explore this idea, DMVZ introduced the *fixed-energy sandpile* model (FES), which involves an explicit free parameter ζ , the density of particles. On a graph with N vertices, the system starts with ζN particles at vertices chosen independently and uniformly at random. Unlike the driven dissipative sandpile described above, there are

no sinks and no addition of particles. Subsequently the system evolves through toppling of unstable sites. Usually the parallel toppling order is chosen: at each time step, all unstable sites topple simultaneously. Toppling may persist forever, or it may stop after some finite time. In the latter case, we say that the system *stabilizes*; in the terminology of DMVZ, it reaches an “absorbing state.”

A common choice of underlying graph for FES is the $n \times n$ square grid with periodic boundary conditions. It is believed, and supported by simulations [20], that there is a *threshold density* ζ_c , such that for $\zeta < \zeta_c$, the system stabilizes with probability tending to 1 as $n \rightarrow \infty$; and for $\zeta > \zeta_c$, with probability tending to 1 the system does not stabilize.

The density conjecture.—For the driven dissipative sandpile on the $n \times n$ square grid with sinks at the boundary, as $n \rightarrow \infty$ the stationary measure has an infinite-volume limit [21], which is a measure on sandpiles on the infinite grid \mathbb{Z}^2 . One gets the same limiting measure whether the grid has periodic or open boundary conditions, and whether there is one sink vertex or the whole boundary serves as a sink [21] (see also [22] for the corresponding result on random spanning trees). The statistical properties of this limiting measure have been much studied [23–25]. Grassberger conjectured that the expected number of particles at a fixed site is $17/8$, and it is now known to be $17/8 \pm 10^{-12}$ [25]. We call this value the *stationary density* ζ_s of \mathbb{Z}^2 .

DMVZ believed that the combination of driving and dissipation in the classical Abelian sandpile model should push it toward the threshold density ζ_c of the fixed-energy sandpile. This leads to a specific testable prediction, which we call the density conjecture.

Density conjecture [4].—On the square grid, $\zeta_c = 17/8$. More generally, $\zeta_c = \zeta_s$.

Vespignani *et al.* [4] write of FES on the square grid, “the system turns out to be critical only for a particular value of the energy density equal to that of the stationary, slowly driven sandpile.” They add that the threshold density ζ_c of the fixed-energy sandpile is “the only possible stationary value for the energy density” of the driven dissipative model. In simulations they find $\zeta_c = 2.1250(5)$, adding in a footnote “It is likely that, in fact, $17/8$ is the exact result.” Other simulations to estimate ζ_c also found the value very close to $17/8$ [1,2].

Our goal in the present Letter is to demonstrate that the density conjecture is more problematic than it first appears. Table I presents data from large-scale simulations indicating that $\zeta_c(\mathbb{Z}^2)$ is 2.125288 to six decimals; close to but not exactly equal to $17/8$. These data are graphed in Fig. 1.

In each trial, we added particles one at a time at uniformly random sites of the $n \times n$ torus. After each addition, we performed topplings until either all sites were stable, or every site toppled at least once. For deterministic sandpiles on a connected graph, if every site topples at least

TABLE I. Fixed-energy sandpile simulations on $n \times n$ tori \mathbb{Z}_n^2 . The third column gives our empirical estimate of the threshold density $\zeta_c(\mathbb{Z}_n^2)$ for \mathbb{Z}_n^2 . The standard deviation in each of our estimates of $\zeta_c(\mathbb{Z}_n^2)$ is 4×10^{-7} . To six decimals, the values of $\zeta_c(\mathbb{Z}_{2048}^2), \dots, \zeta_c(\mathbb{Z}_{16384}^2)$ are all the same. The rapid convergence is due in part to periodic boundary conditions.

n	Trials	Estimate of $\zeta_c(\mathbb{Z}_n^2)$
64	2^{28}	$2.124\,956\,1 \pm 0.000\,000\,4$
128	2^{26}	$2.125\,185\,1 \pm 0.000\,000\,4$
256	2^{24}	$2.125\,257\,2 \pm 0.000\,000\,4$
512	2^{22}	$2.125\,278\,6 \pm 0.000\,000\,4$
1024	2^{20}	$2.125\,285\,3 \pm 0.000\,000\,4$
2048	2^{18}	$2.125\,287\,6 \pm 0.000\,000\,4$
4096	2^{16}	$2.125\,287\,7 \pm 0.000\,000\,4$
8192	2^{14}	$2.125\,288\,0 \pm 0.000\,000\,4$
16384	2^{12}	$2.125\,287\,7 \pm 0.000\,000\,4$

once, the system will never stabilize [26–28]. We recorded m/n^2 as an empirical estimate of the threshold density $\zeta_c(\mathbb{Z}_n^2)$, where m was the maximum number of particles for which the system stabilized. We averaged these empirical estimates over many independent trials.

We used a random number generator based on the Advanced Encryption Standard (AES-256), which has been found to exhibit excellent statistical properties [29,30]. Our simulations were conducted on a high performance computing (HPC) cluster of computers.

Phase transition at ζ_c threshold.—We consider the density conjecture on several other families of graphs, including some for which we can determine the exact values ζ_c and ζ_s analytically.

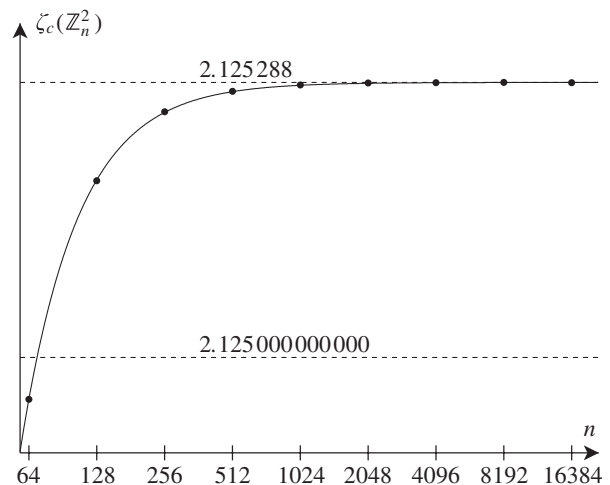


FIG. 1. The data of Table I from $n = 64$ to $n = 16384$ are well approximated by $\zeta_c(\mathbb{Z}_n^2) = 2.1252881 \pm 3 \times 10^{-7} - (0.390 \pm 0.001)n^{-1.7}$. (The error bars are too small to be visible, so the data are shown as points.) We conclude that the asymptotic threshold density $\zeta_c(\mathbb{Z}_n^2)$ is 2.125288 to six decimals. In contrast, the stationary density $\zeta_s(\mathbb{Z}_n^2)$ is 2.125000000000 to twelve decimals.

Dhar [12] defined recurrent sandpile configurations and showed that they form an Abelian group. A consequence of his result is that the stationary measure for the driven dissipative sandpile on a finite graph G with sinks is the uniform measure on recurrent configurations. The *stationary density* $\zeta_s(G)$ is the expected total number of particles in a uniform random recurrent configuration, divided by the number of nonsink vertices in G .

The threshold density ζ_c and stationary density ζ_s for different graphs is summarized in Table II. The only graph on which the two densities are known to be equal is \mathbb{Z} [17,18,27]. On all other graphs we examined, with the possible exception of the 3-regular Cayley tree, it appears that $\zeta_c \neq \zeta_s$.

Each row of Table II represents an infinite family of graphs G_n indexed by an integer $n \geq 1$. For example, for \mathbb{Z}^2 we take G_n to be the $n \times n$ square grid, and for the regular trees we take G_n to be a finite tree of depth n . As sinks in G_n we take the set of boundary sites $G_n \setminus G_{n-1}$ (note that on trees this corresponds to wired boundary conditions). The value of ζ_s reported is $\lim_{n \rightarrow \infty} \zeta_s(G_n)$.

The exact values of ζ_s for regular trees (Bethe lattices) were calculated by Dhar and Majumdar [31]. The corresponding values of ζ_c we report come from simulations [32]. We derive or simulate the values of ζ_s and ζ_c for the bracelet, flower, ladder, and complete graphs in [32].

As an example, consider the *bracelet graph* B_n , which is a cycle of n vertices, with each edge doubled (see Fig. 2). A site topples by sending out 4 particles: 2 to each of its two neighbors. One site serves as the sink. To compare the densities ζ_c and ζ_s , we consider the driven dissipative sandpile before it reaches stationarity, by running it for time λ . More precisely, we place λn particles uniformly at random, stabilize the resulting sandpile, and let $\rho_n(\lambda)$ denote the expected density of the resulting stable configuration. In the long version of this Letter [32] we prove the following:

Theorem 1 ([32]).—For the bracelet graph B_n , in the limit as $n \rightarrow \infty$, (i) The threshold density ζ_c is the unique positive root of $\zeta = \frac{5}{2} - \frac{1}{2}e^{-2\zeta}$ (numerically, $\zeta_c = 2.496608$). (ii) The stationary density ζ_s is $5/2$. (iii) The final density $\rho_n(\lambda)$, as a function of initial density λ , converges pointwise to a limit $\rho(\lambda)$, where

$$\rho(\lambda) = \min\left(\lambda, \frac{5 - e^{-2\lambda}}{2}\right) = \begin{cases} \lambda, & \lambda \leq \zeta_c \\ \frac{5 - e^{-2\lambda}}{2}, & \lambda > \zeta_c. \end{cases}$$

Part 3 of this theorem shows that despite the inequality $\zeta_s \neq \zeta_c$, a connection remains between the driven dissipative dynamics used to define ζ_s and the conservative dynamics used to define ζ_c : since the derivative $\rho'(\lambda)$ is discontinuous at $\lambda = \zeta_c$, the driven sandpile undergoes a second-order phase transition at density ζ_c . For $\lambda < \zeta_c$, the driven sandpile loses very few particles to the sink, and the final density equals the initial density λ ; for $\lambda > \zeta_c$ it loses a macroscopic proportion of particles to the sink, so the

TABLE II. Stationary and threshold densities for different graphs. Exact values are in bold.

Graph	ζ_s	ζ_c
\mathbb{Z}	1	1
\mathbb{Z}^2	17/8 = 2.125	2.125288...
Bracelet	5/2 = 2.5	2.496608...
Flower graph	5/3 = 1.666667...	1.668898...
Ladder graph	$\frac{7}{4} - \frac{\sqrt{3}}{12} = 1.605662...$	1.6082...
Complete graph	$1/2 \times n + O(\sqrt{n})$	$1 \times n - O(\sqrt{n \log n})$
3-regular tree	3/2	1.50000...
4-regular tree	2	2.00041...
5-regular tree	5/2	2.51167...

final density is strictly smaller than λ . As Fig. 3 shows, the sandpile continues to evolve as λ increases beyond ζ_c ; in particular, its density keeps increasing.

We are also able to prove that a similar phase transition occurs on the *flower graph*, shown in Fig. 2. Interestingly, the final density $\rho(\lambda)$ for the flower graph is a *decreasing* function of $\lambda > \zeta_c$ (Fig. 3 bottom).

Our proofs make use of local toppling invariants on these graphs. On the bracelet graph, since particles always travel in pairs, the parity of the number of particles on a single vertex is conserved. On the flower graph, the differ-

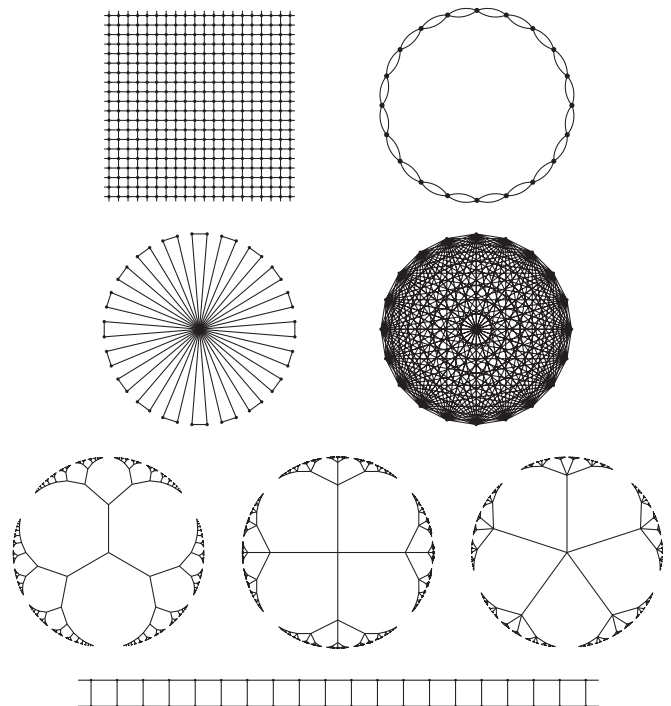


FIG. 2. The graphs on which we compare ζ_s and ζ_c : the grid (upper left), bracelet graph (upper right), flower graph (2nd row left), complete graph (2nd row right), Cayley trees (Bethe lattices) of degree $d = 3, 4, 5$ (3rd row), and ladder graph (bottom).

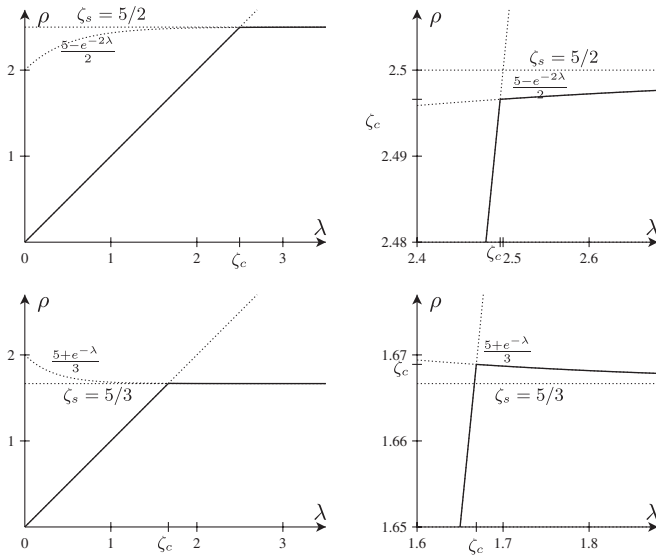


FIG. 3. Density $\rho(\lambda)$ of the final stable configuration as a function of initial density λ on the bracelet graph (top row) and flower graph (bottom row) as the graph size tends to infinity. A phase transition occurs at $\lambda = \zeta_c$. At first glance (left panels) it appears that the driven sandpile reaches its stationary density ζ_s at this point, but closer inspection (right panels) reveals that the final density $\rho(\lambda)$ continues to change as λ increases beyond ζ_c . These graphs are exact.

ence modulo 3 of the number of particles on the two vertices in a single “petal” is conserved.

One might guess that the failure of the density conjecture is due only to the existence of local toppling invariants, or else to boundary effects from the sinks. The ladder graph (Fig. 2) has no local toppling invariants; moreover, it is essentially one dimensional, so the bulk of the graph is well insulated from the sinks at the boundary. Nevertheless, we find [32] a small but undeniable difference between ζ_s and ζ_c on the ladder graph.

Conclusions.—The conclusion of [5] that “FES are shown to exhibit an absorbing state transition with critical properties coinciding with those of the corresponding sandpile model” should be reevaluated.

In response to this Letter, several researchers have suggested to us that perhaps the density conjecture holds for stochastic sandpiles even if not for deterministic ones. This hypothesis deserves some scrutiny.

For the driven dissipative sandpile, there is a transition point at the threshold density of the FES, beyond which a macroscopic amount of sand begins to dissipate. The continued evolution of the sandpile beyond ζ_c shows that driven sandpiles have (at least) a one-parameter family of distinct critical states. While the stationary state has rightly been the object of intense study, we suggest that these additional critical states deserve further attention.

- [1] R. Dickman, A. Vespignani, and S. Zapperi, Phys. Rev. E **57**, 5095 (1998).
- [2] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Phys. Rev. Lett. **81**, 5676 (1998).
- [3] R. Dickman, M. Muñoz, A. Vespignani, and S. Zapperi, Braz. J. Phys. **30**, 27 (2000).
- [4] A. Vespignani, R. Dickman, M. A. Muñoz, and S. Zapperi, Phys. Rev. E **62**, 4564 (2000).
- [5] M. A. Muñoz, R. Dickman, R. Pastor-Satorras, A. Vespignani, and S. Zapperi, in *Proceedings of the 6th Granada Seminar on Computational Physics* edited by J. Marro and P.L. Garrido (AIP, Melville, NY, 2001).
- [6] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
- [7] P. Grassberger and S. S. Manna, J. Phys. (Les Ulis, Fr.) **51**, 1077 (1990).
- [8] J. A. Bonachela and M. A. Muñoz, Phys. Rev. E **78**, 041102 (2008).
- [9] R. Vidigal and R. Dickman, J. Stat. Phys. **118** 1 (2005).
- [10] S. D. da Cunha, R. R. Vidigal, L. R. da Silva, and R. Dickman, arXiv:0906.1392.
- [11] F. Redig, Les Houches lecture notes (2005).
- [12] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
- [13] D. V. Kvitarev, S. Lübeck, P. Grassberger, and V. B. Priezhev, Phys. Rev. E **61**, 81 (2000).
- [14] M. DeMenech, A. L. Stella, and C. Tebaldi, Phys. Rev. E **58**, R2677 (1998).
- [15] R. Karmakar, S. S. Manna, and A. L. Stella, Phys. Rev. Lett. **94**, 088002 (2005).
- [16] D. Sornette, *Critical Phenomena in Natural Sciences* (Springer-Verlag, Berlin, 2004), 2nd ed.
- [17] R. Meester and C. Quant, Markov Processes Relat. Fields **11**, 355 (2005).
- [18] A. Fey-den Boer and F. Redig, Markov Processes Relat. Fields **11**, 425 (2005).
- [19] L. T. Rolla and V. Sidoravicius, arXiv:0908.1152.
- [20] F. Bagnoli, F. Cecconi, A. Flammini, and A. Vespignani, Europhys. Lett. **63**, 512 (2003).
- [21] S. R. Athreya and A. A. Járai, Commun. Math. Phys. **249**, 197 (2004).
- [22] R. Pemantle, Ann. Probab. **19**, 1559 (1991).
- [23] S. N. Majumdar and D. Dhar, J. Phys. A **24**, L357 (1991).
- [24] V. B. Priezhev, J. Stat. Phys. **74**, 955 (1994).
- [25] M. Jeng, G. Piroux, and P. Ruelle, J. Stat. Mech. (2006) P10015.
- [26] G. Tardos, SIAM J. Discrete Math. **1**, 397 (1988).
- [27] A. Fey-den Boer, R. Meester, and F. Redig, Ann. Probab. **37**, 654 (2009).
- [28] A. Fey, L. Levine, and Y. Peres, J. Stat. Phys. **138**, 143 (2010).
- [29] P. Hellekalek and S. Wegenkittl, ACM Trans. Model. Comput. Simul. **13**, 322 (2003).
- [30] P. L’Ecuyer and R. Simard, ACM Trans. Math. Softw. **33**, No. 22, 40 (2007).
- [31] D. Dhar and S. N. Majumdar, J. Phys. A **23**, 4333 (1990).
- [32] A. Fey, L. Levine, and D. B. Wilson, arXiv:1001.3401.