Adler Function, Bjorken Sum Rule, and the Crewther Relation to Order α_s^4 in a General Gauge Theory

P. A. Baikov
Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119991, Russia

K. G. Chetyrkin* and J. H. Kühn
Institut für Theoretische Teilchenphysik, KIT, D-76128 Karlsruhe, Germany
(Received 20 January 2010; published 31 March 2010)

We compute, for the first time, the order α_s^4 contributions to the Bjorken sum rule for polarized electron-nucleon scattering and to the (nonsinglet) Adler function for the case of a generic color gauge group. We confirm at the same order a (generalized) Crewther relation which provides a strong test of the correctness of our previously obtained results: the QCD Adler function and the five-loop β function in quenched QED. In particular, the appearance of an irrational contribution proportional to ζ_3 in the latter quantity is confirmed. We obtain the *commensurate* scale equation relating the effective strong coupling constants as inferred from the Bjorken sum rule and from the Adler function at order α_s^4.

DOI: 10.1103/PhysRevLett.104.132004 PACS numbers: 12.38.Bx, 12.20.−m

Introduction.—The Crewther relation [1,2] relates in a nontrivial way two seemingly disconnected quantities, namely, the (nonsinglet) Adler function [3] D and the coefficient function C^{Bjp}, describing the deviation of the Bjorken sum rule [4] for polarized deep inelastic scattering from its naive-parton model value. The Adler function is defined through the correlator of the vector current j_μ

$$3Q^2\Pi(Q^2) = i\int d^4x e^{iq\cdot x}\langle 0|Tj_\mu(x)j^\mu(0)|0\rangle, \qquad (1)$$

as follows

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2), \qquad (2)$$

with $Q^2 = -q^2$. In fact, the Adler function is the main theoretical object required to study such important physical observables as the cross section for electron-positron annihilation into hadrons and the hadronic decay rates of both the Z boson and the τ lepton (see, e.g., [5]). The Bjorken sum rule expresses the integral over the spin distributions of quarks inside of the nucleon in terms of its axial charge times a coefficient function C^{Bjp},

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 [g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)]dx = \frac{g_A}{6}C^{Bjp}(a_s) + \sum_{i=2}^{\infty}\frac{\mu_{2i}^{p-n}(Q^2)}{Q^{2i-2}}, \qquad (3)$$

where g_1^{ep} and g_1^{en} are the spin-dependent proton and neutron structure functions, g_A is the nucleon axial charge as measured in neutron β decay. The coefficient function $C^{Bjp}(a_s) = 1 + \mathcal{O}(a_s)$ is proportional to the flavor-nonsinglet axial vector current $\bar\psi\gamma^\mu\gamma_5\psi$ in the corresponding short distance Wilson expansion. The sum in the second line of (3) describes for the nonperturbative power corrections (higher twist) which are inaccessible for pQCD. Within perturbative QCD, we define

$$D(Q^2) = d_R\Big(1 + \frac{3}{4}C_F a_s + \sum_{i=2}^{\infty} d_i a_s^i(Q^2)\Big),$$

$$C^{Bjp}(Q^2) = 1 - \frac{3}{4}C_F a_s + \sum_{i=2}^{\infty} c_i a_s^i(Q^2),$$

$$1/C^{Bjp}(Q^2) = 1 + \frac{3}{4}C_F a_s + \sum_{i=2}^{\infty} b_i a_s^i(Q^2),$$

where d_R is the dimension of the quark color representation (for QCD $d_R = 3$), $a_s \equiv \alpha_s/\pi$, and the normalization scale μ is set $\mu^2 = Q^2$. Note that we consider only the so-called "nonsinglet" contribution to the Adler function and do not write explicitly a common factor $\sum_i Q_i^2$ (with Q_i being the electric charge of the i-th quark flavor) for $R(s)$.

The Crewther relation states that

$$D(a_s)C^{Bjp}(a_s) = d_R\Big[1 + \frac{\beta(a_s)}{a_s}K(a_s)\Big], \qquad K(a_s) = K_0 + a_s K_1 + a_s^2 K_2 + a_s^3 K_3 + \dots. \qquad (4)$$

Here, $\beta(a_s) = \mu^2\frac{d}{d\mu^2}a_s(\mu) = -\sum_{i\geq 0}\beta_i a_s^{i+2}$ is the QCD β function describing the *running* of the coupling constant a_s with respect to a change of the normalization scale μ and with its first term $\beta_0 = \frac{11}{12}C_A - \frac{T}{3}n_f$ being responsible for asymptotic freedom of QCD. The term proportional to the β function describes the deviation from the limit of exact conformal invariance, with the deviations starting in order α_s^2, and was suggested [2] on the basis of $\mathcal{O}(\alpha_s^3)$ calculations of $D(a_s)$ [6,7] and $C^{Bjp}(a_s)$ [8]. A formal proof was carried out in [9,10]. The original relation without this term was first proposed in [1] (see, also, [11]).

At order α_s, the Crewther relation is evidently fulfilled. The color structures which appear in d_n and c_n (hence, also in b_n) for $n = 1$, 2, 3, and 4 are

$$a_s^1\text{: } C_F, \qquad a_s^2\text{: } C_F^2, C_FT_f, C_FC_A,$$
$$a_s^3\text{: } C_F^3, C_F^2T_f, C_FT_f^2, C_F^2C_A, C_FT_fC_A, C_FC_A^2,$$
$$a_s^4\text{: } \frac{d_F^{abcd}d_A^{abcd}}{d_R}, \frac{n_fd_F^{abcd}d_F^{abcd}}{d_R}, C_F^4, C_F^3T_f, C_F^2T_f^2, C_FT_f^3, C_F^3C_A, C_F^2T_fC_A, C_FT_f^2C_A, C_F^2C_A^2, C_FT_fC_A^2, C_FC_A^3. \quad (5)$$

Here, C_F and C_A are the quadratic Casimir operators of the fundamental and the adjoint representation of the Lie algebra, T is the trace normalization of the fundamental representation, $T_f \equiv Tn_f$, with n_f being the number of quark flavors. The exact definitions of $d_F^{abcd}d_A^{abcd}$ and $d_F^{abcd}d_F^{abcd}$ are given in [12]. For QCD (color gauge group $SU(3)$),

$$C_F = 4/3, \qquad C_A = 3, \qquad T = 1/2, \qquad d_R = 3,$$
$$d_F^{abcd}d_A^{abcd} = \frac{15}{2}, \qquad d_F^{abcd}d_F^{abcd} = \frac{5}{12}. \quad (6)$$

Note, that all color structures, apart from the d^2 terms which appear first at order α_s^4, involve at least one factor C_F. As a consequence, K_0 must be set to zero. An inspection of Eqs. (4) and (5) clearly shows that the color structures which may appear in a coefficient K_i are identical to those appearing in the coefficient b_{i-1} and c_{i-1}, listed in Eq. (5). Thus, at orders α_s^2, α_s^3, and α_s^4, the Crewther relation puts as many as 2, 3, and, finally, 6 constraints on the differences $d_2 - b_2$, $d_3 - b_3$, and $d_4 - b_4$, respectively. The fulfillment of these constraints constitutes a powerful check of the correctness of the calculations of $D^{NS}(a_s)$ and $C^{Bjp}(a_s)$.

Indeed, at orders $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_s^3)$, the results for $D^{NS}(a_s)$ and $1/C^{Bjp}(a_s)$

$$d_2 = -\frac{3}{32}C_F^2 + C_FT_f\left[\zeta_3 - \frac{11}{8}\right] + C_FC_A\left[\frac{123}{32} - \frac{11\zeta_3}{4}\right],$$
$$b_2 = -\frac{3}{32}C_F^2 + C_FT_f\left[-\frac{1}{2}\right] + C_FC_A\left[\frac{23}{16}\right],$$
$$d_3 = -\frac{69}{128}C_F^3 + C_F^2T_f\left[-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right] + C_FT_f^2\left[\frac{151}{54} - \frac{19}{9}\zeta_3\right] + C_F^2C_A\left[-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right] + C_FT_fC_A\left[-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_FC_A^2\left[\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right],$$
$$b_3 = -\frac{69}{128}C_F^3 + C_F^2T_f\left[-\frac{299}{576} + \frac{5}{12}\zeta_3\right] + C_FT_f^2\left[\frac{115}{216}\right] + C_F^2C_A\left[\frac{1}{576} + \frac{11}{12}\zeta_3\right] + C_FT_fC_A\left[-\frac{3535}{864} - \frac{3}{4}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_FC_A^2\left[\frac{5437}{864} - \frac{55}{24}\zeta_5\right]$$

are well consistent [2] with all 5 constraints on the coefficients d_2, d_3, b_2, and b_3 and imply

$$K_1 = C_F\left(-\frac{21}{8} + 3\zeta_3\right),$$
$$K_2 = C_FT_f\left(\frac{163}{24} - \frac{19}{3}\zeta_3\right) + C_FC_A\left(-\frac{629}{32} + \frac{221}{12}\zeta_3\right) + C_F^2\left(\frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5\right).$$

The next, $\mathcal{O}(\alpha_s^4)$, contribution to $D(a_s)$ has been recently computed [13] for QCD, i.e., setting the color structures to their $SU(3)$ numerical values [Eq. (6)]. The function $C^{Bjp}(a_s)$ is known to order α_s^3 *only*.

The importance of computation of the $\mathcal{O}(\alpha_s^4)$ contribution to the both coefficients d_4 and b_4 for a generic color gauge group comes from a few reasons.

First, the knowledge of c_4 in the Bjorken sum rule is vital for proper extraction of higher twist contributions. Indeed, in [14], the recent Jefferson Lab data on the spin-dependent proton and neutron structure functions [15–19] were used to extract the leading and subleading higher twist parameters μ_4 and μ_6. It has been demonstrated that, say, the twist four term μ_4 approximately halves its value in transition from LO to NLO, and from NLO to NNLO. This *duality* between perturbative and nonperturbative contributions has been observed before for the structure function F_3 [20] (for a related recent discussion, see also [21]).

Second, the Bjorken sum rule provides us with a very convenient definition of the *effective strong coupling constant* (ECC) [19,22], namely,

$$6\Gamma_1^{p-n}(Q^2) = g_A[1 - a_{g_1}(Q^2)]. \quad (7)$$

This quantity is directly measurable down to vanishing values of Q^2 and, due to Eq. (3), approaches to the standard $\alpha_s(Q)$ at large Q^2. It is by definition gauge and scheme invariant. Another convenient ECC, a_D, comes from the Adler function [23]

$$D(Q^2) = 1 + a_D(Q^2). \quad (8)$$

As its perturbative expansion is available to $\mathcal{O}(\alpha_s^4)$ [13], the knowledge of c_4 will allow for the first time to compare two ECC's with the help of a commensurate scale relation (CSR) [24] at an order unprecedented to date.

Third, the six constraints imposed by Eq. (4) provide a highly nontrivial and welcome check of the calculation of d_4 in QCD [13]. In particular, in [25], we computed a part of the full result for d_4, namely, the term proportional to the color structure C_F^4. As is well known, an interesting object—the β function of quenched QED—can be inferred from the part of the Adler function which depends on C_F only by setting $C_F = 1$ and adjusting a global normalization factor. The result ($A \equiv \frac{\alpha}{4\pi}$)

$$\beta^{q\mathrm{QED}} = \frac{4}{3}A + 4A^2 - 2A^3 - 46A^4 + \left(\frac{4157}{6} + 128\zeta_3\right)A^5 \quad (9)$$

revealed an unexpected [26] appearance of the irrational constant ζ_3 at five loops and cast doubt on the correctness of the full QCD result for d_4 [29].

Using the same techniques as in calculations of [8,13], we have computed the Adler function and the function C^{Bjp} for a general gauge group to order α_s^4. Our results read

$$\begin{aligned}
d_4 = {} & \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right] + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right] + C_F^4\left[\frac{4157}{2048} + \frac{3}{8}\zeta_3\right] \\
& + C_F^3T_f\left[\frac{1001}{384} + \frac{99}{32}\zeta_3 - \frac{125}{4}\zeta_5 + \frac{105}{4}\zeta_7\right] + C_F^2T_f^2\left[\frac{5713}{1728} - \frac{581}{24}\zeta_3 + \frac{125}{6}\zeta_5 + 3\zeta_3^2\right] + C_FT_f^3\left[-\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5\right] \\
& + C_F^3C_A\left[-\frac{253}{32} - \frac{139}{128}\zeta_3 + \frac{2255}{32}\zeta_5 - \frac{1155}{16}\zeta_7\right] + C_F^2T_fC_A\left[\frac{32357}{13824} + \frac{10661}{96}\zeta_3 - \frac{5155}{48}\zeta_5 - \frac{33}{4}\zeta_3^2 - \frac{105}{8}\zeta_7\right] \\
& + C_FT_f^2C_A\left[\frac{340843}{5184} - \frac{10453}{288}\zeta_3 - \frac{170}{9}\zeta_5 - \frac{1}{2}\zeta_3^2\right] + C_F^2C_A^2\left[-\frac{592141}{18432} - \frac{43925}{384}\zeta_3 + \frac{6505}{48}\zeta_5 + \frac{1155}{32}\zeta_7\right] \\
& + C_FT_fC_A^2\left[-\frac{4379861}{20736} + \frac{8609}{72}\zeta_3 + \frac{18805}{288}\zeta_5 - \frac{11}{2}\zeta_3^2 + \frac{35}{16}\zeta_7\right] \\
& + C_FC_A^3\left[\frac{52207039}{248832} - \frac{456223}{3456}\zeta_3 - \frac{77995}{1152}\zeta_5 + \frac{605}{32}\zeta_3^2 - \frac{385}{64}\zeta_7\right],
\end{aligned} \quad (10)$$

$$\begin{aligned}
b_4 = {} & \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left[\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right] + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left[-\frac{13}{16} - \zeta_3 + \frac{5}{2}\zeta_5\right] + C_F^4\left[\frac{4157}{2048} + \frac{3}{8}\zeta_3\right] \\
& + C_F^3T_f\left[-\frac{473}{2304} - \frac{391}{96}\zeta_3 + \frac{145}{24}\zeta_5\right] + C_F^2T_f^2\left[\frac{869}{576} - \frac{29}{24}\zeta_3\right] + C_FT_f^3\left[-\frac{605}{972}\right] + C_F^3C_A\left[-\frac{8701}{4608} + \frac{1103}{96}\zeta_3 - \frac{1045}{48}\zeta_5\right] \\
& + C_F^2T_fC_A\left[-\frac{17309}{13824} + \frac{1127}{144}\zeta_3 - \frac{95}{144}\zeta_5 - \frac{35}{4}\zeta_7\right] + C_FT_f^2C_A\left[\frac{165283}{20736} + \frac{43}{144}\zeta_3 - \frac{5}{12}\zeta_5 + \frac{1}{6}\zeta_3^2\right] \\
& + C_F^2C_A^2\left[-\frac{435425}{55296} - \frac{1591}{144}\zeta_3 + \frac{55}{9}\zeta_5 + \frac{385}{16}\zeta_7\right] + C_FT_fC_A^2\left[-\frac{1238827}{41472} - \frac{59}{64}\zeta_3 + \frac{1855}{288}\zeta_5 - \frac{11}{12}\zeta_3^2 + \frac{35}{16}\zeta_7\right] \\
& + C_FC_A^3\left[\frac{8004277}{248832} - \frac{1069}{576}\zeta_3 - \frac{12545}{1152}\zeta_5 + \frac{121}{96}\zeta_3^2 - \frac{385}{64}\zeta_7\right].
\end{aligned} \quad (11)$$

All six constraints from the generalized Crewther relation are indeed met with

$$\begin{aligned}
K_3 = {} & C_F^3\left(\frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7\right) + C_F^2T_f\left(-\frac{7729}{1152} - \frac{917}{16}\zeta_3 + \frac{125}{2}\zeta_5 + 9\zeta_3^2\right) + C_FT_f^2\left(-\frac{307}{18} + \frac{203}{18}\zeta_3 + 5\zeta_5\right) \\
& + C_F^2C_A\left(\frac{99757}{2304} + \frac{8285}{96}\zeta_3 - \frac{1555}{12}\zeta_5 - \frac{105}{8}\zeta_7\right) + C_FT_fC_A\left(\frac{1055}{9} - \frac{2521}{36}\zeta_3 - \frac{125}{3}\zeta_5 - 2\zeta_3^2\right) \\
& + C_FC_A^2\left(-\frac{406043}{2304} + \frac{18007}{144}\zeta_3 + \frac{2975}{48}\zeta_5 - \frac{77}{4}\zeta_3^2\right).
\end{aligned}$$

Note that coefficients in front of first three color structures in Eqs. (10) and (11), (C_F^4, $n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}$ and $\frac{d_F^{abcd}d_A^{abcd}}{d_R}$) are equal, as they should be. The C_F^4 term, in particular, provides us with a beautiful confirmation of the correctness of the result (9) for the qQED β function (the test was originally suggested in [29]).

It is interesting to note that the results do not depend on ζ_n with $n = 2, 4, 6$. Also, an unexpected feature of our results is the *separate* proportionality all terms of highest and subhighest transcendentality in a given loop order (that is ζ_3^2 and ζ_7 at α_s^4, ζ_5 at α_s^3 and, at last, ζ_3 at α_s^2) to β_0. This feature *is not* required by (4), the latter essentially constraints only the *difference* $d_i - b_i$.

In numerical form, C^{Bjp} reads (with all color factors set to their QCD values)

$$\begin{aligned}
C^{Bjp} = {} & 1 - a_s + (-4.583 + 0.3333n_f)a_s^2 \\
& + a_s^3(-41.44 + 7.607n_f - 0.1775n_f^2)a_s^3 \\
& + (-479.4 + 123.4n_f - 7.697n_f^2 + 0.1037n_f^3)a_s^4.
\end{aligned} \quad (12)$$

It is of interest to compare the newly found coefficient in front of the α_s^4 term with well-known predictions [30]

$$c_4^{\mathrm{pred}}(n_f = 3, 4, 5, 6) = -130, -58, -18, 22$$

and

$$c_4^{\text{exact}}(n_f = 3, 4, 5, 6) = -175.7, -102.4, -41.96, 6.2.$$

At last, we derive the CSR connecting the two ECC's a_{g_1} and a_D as defined in Eqs. (7) and (8). Following Ref. [31], we get for QCD

$$[1 + a_D(Q^{\star 2})][1 - a_{g_1}(Q^2)] = 1, \qquad (13)$$

with $[a_D^\star = a_D(Q^{\star 2})]$

$$\begin{aligned}\ln\left(\frac{Q^{\star 2}}{Q^2}\right) &= -K_1 + a_D^\star[\beta_0 K_1^2 + 2d_2 K_1 - K_1 - K_2] \\ &\quad + (a_D^\star)^2\bigg[\beta_0(-6d_2K_1^2 + 2K_1^2 + 3K_2K_1) \\ &\quad - 2\beta_0^2 K_1^3 + K_1\left(\frac{3}{2}\beta_1 K_1 - 6d_2^2 + 2d_2 + 3d_3\right) \\ &\quad + K_2(3d_2 - 1) - K_3\bigg] \\ &= -1.30823 + a_D^\star[0.80241 - 0.03933 n_f] \\ &\quad + (a_D^\star)^2[-16.9020 + 2.62311 n_f \\ &\quad - 0.10202 n_f^2].\end{aligned}$$

Let us emphasize that Eq. (13) constitutes a definitive and precise prediction of QCD. Relating essentially two observables, it is devoid of scale and scheme ambiguities. (For more details on CRS's, see, e.g., [32].)

In conclusion, we want to mention that all our calculations have been performed on a SGI ALTIX 24-node IB-interconnected cluster of 8-cores Xeon computers and on the HP XC4000 supercomputer of the federal state Baden-Württemberg using parallel MPI-based [33] as well as thread-based [34] versions of FORM [35]. For evaluation of color factors, we have used the FORM program COLOR [36]. The diagrams have been generated with QGRAF [37].

This work was supported by the Deutsche Forschungsgemeinschaft in the Sonderforschungsbereich/Transregio SFB/TR-9 "Computational Particle Physics" and by RFBR Grant No. 08-02-01451. We thank V. M. Braun for useful discussions.

*Permanent address: Institute for Nuclear Research, Russian Academy of Sciences, Moscow 117312, Russia.

[1] R. J. Crewther, Phys. Rev. Lett. **28**, 1421 (1972).
[2] D. J. Broadhurst and A. L. Kataev, Phys. Lett. B **315**, 179 (1993).
[3] S. L. Adler, Phys. Rev. D **10**, 3714 (1974).
[4] J. D. Bjorken, Phys. Rev. **163**, 1767 (1967); Phys. Rev. D **1**, 1376 (1970).
[5] K. G. Chetyrkin, J. H. Kuhn, and A. Kwiatkowski, Phys. Rep. **277**, 189 (1996).
[6] S. G. Gorishnii, A. L. Kataev, and S. A. Larin, Phys. Lett. B **259**, 144 (1991).
[7] L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66**, 560 (1991).
[8] S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **259**, 345 (1991).
[9] R. J. Crewther, Phys. Lett. B **397**, 137 (1997).
[10] V. M. Braun, G. P. Korchemsky, and D. Mueller, Prog. Part. Nucl. Phys. **51**, 311 (2003).
[11] S. L. Adler, C. G. Callan, Jr., D. J. Gross, and R. Jackiw, Phys. Rev. D **6**, 2982 (1972).
[12] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. B **405**, 327 (1997).
[13] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, Phys. Rev. Lett. **101**, 012002 (2008).
[14] R. S. Pasechnik, D. V. Shirkov, and O. V. Teryaev, Phys. Rev. D **78**, 071902(R) (2008).
[15] P. E. Bosted *et al.* (CLAS Collaboration), Phys. Rev. C **75**, 035203 (2007).
[16] Y. Prok *et al.* (CLAS Collaboration), Phys. Lett. B **672**, 12 (2009).
[17] A. Deur *et al.*, Phys. Rev. D **78**, 032001 (2008).
[18] A. Deur *et al.*, Phys. Rev. Lett. **93**, 212001 (2004).
[19] A. Deur, V. Burkert, J.-P. Chen, and W. Korsch, Phys. Lett. B **650**, 244 (2007).
[20] A. L. Kataev, G. Parente, and A. V. Sidorov, Nucl. Phys. **B573**, 405 (2000).
[21] S. Narison and V. I. Zakharov, Phys. Lett. B **679**, 355 (2009).
[22] G. Grunberg, Phys. Lett. B **95**, 70 (1980).
[23] S. J. Brodsky, S. Menke, C. Merino, and J. Rathsman, Phys. Rev. D **67**, 055008 (2003).
[24] S. J. Brodsky and H. J. Lu, Phys. Rev. D **51**, 3652 (1995).
[25] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Proc. Sci., RADCOR2007 (2007) 023 [arXiv:0810.4048].
[26] Unexpected, because there existed a widespread belief that the rationality property is not accidental but holds also in higher orders [27,28].
[27] C. M. Bender, R. W. Keener, and R. E. Zippel, Phys. Rev. D **15**, 1572 (1977).
[28] D. J. Broadhurst, arXiv:hep-th/9909185.
[29] A. L. Kataev, Phys. Lett. B **668**, 350 (2008).
[30] A. L. Kataev and V. V. Starshenko, Mod. Phys. Lett. A **10**, 235 (1995).
[31] S. J. Brodsky, G. T. Gabadadze, A. L. Kataev, and H. J. Lu, Phys. Lett. B **372**, 133 (1996).
[32] S. J. Brodsky and J. Rathsman, arXiv:hep-ph/9906339.
[33] M. Tentyukov *et al.*, arXiv:cs/0407066.
[34] M. Tentyukov and J. A. M. Vermaseren, arXiv:hep-ph/0702279.
[35] J. A. M. Vermaseren, arXiv:math-ph/0010025.
[36] T. van Ritbergen, A. N. Schellekens, and J. A. M. Vermaseren, Int. J. Mod. Phys. A **14**, 41 (1999).
[37] P. Nogueira, J. Comput. Phys. **105**, 279 (1993).